

Dedicated to the memory of
professor N.M. Korobov.

A reinforcement of the Bourgain-Kontorovich's theorem.

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Аннотация

Zaremba's conjecture (1971) states that every positive integer number d can be represented as a denominator (continuant) of a finite continued fraction $\frac{b}{d} = [d_1, d_2, \dots, d_k]$, with all partial quotients d_1, d_2, \dots, d_k being bounded by an absolute constant A . Recently (in 2011) several new theorems concerning this conjecture were proved by Bourgain and Kontorovich. The easiest of them states that the set of numbers satisfying Zaremba's conjecture with $A = 50$ has positive proportion in \mathbb{N} . In this paper the same theorem is proved with $A = 7$.

Bibliography: 15 titles.

Keywords: continued fraction, continuant, exponential sums.

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Chapter I

Introduction

1 Historical background

Let \mathfrak{R}_A be the set of rational numbers whose continued fraction expansion has all partial quotients being bounded by A :

$$\mathfrak{R}_A = \left\{ \frac{b}{d} = [d_1, d_2, \dots, d_k] \mid 1 \leq d_j \leq A \text{ для } j = 1, \dots, k \right\},$$

where

$$[d_1, \dots, d_k] = \frac{1}{d_1 + \cfrac{1}{\ddots + \cfrac{1}{d_k}}} \quad (1.1)$$

is a finite continued fraction, d_1, \dots, d_k are partial quotients. Let \mathfrak{D}_A be the set of denominators of numbers in \mathfrak{R}_A :

$$\mathfrak{D}_A = \left\{ d \mid \exists b : (b, d) = 1, \frac{b}{d} \in \mathfrak{R}_A \right\}.$$

And set

$$\mathfrak{D}_A(N) = \left\{ d \in \mathfrak{D}_A \mid d \leq N \right\}.$$

Conjecture 1.1. (*Zaremba's conjecture* [12, p. 76], 1971). For sufficiently large A one has

$$\mathfrak{D}_A = \mathbb{N}.$$

That is, every $d \geq 1$ can be represented as a denominator of a finite continued fraction $\frac{b}{d}$ whose partial quotients are bounded by A . In fact Zaremba conjectured that $A = 5$ is already large enough. A bit earlier, in the 1950-th, studying problems concerning numerical integration Bahvalov, Chensov and N.M. Korobov also made the same assumption. But they did not publish it anywhere. Korobov [15] proved that for a prime p there exists a , such that the greatest partial quotient of $\frac{a}{p}$ is smaller than $\log p$. A detailed survey on results concerning Zaremba's conjecture can be found in [1], [11].

Bourgain and Kontorovich suggested that the problem should be generalized in the following way. Let $\mathcal{A} \in \mathbb{N}$ be any finite alphabet ($|\mathcal{A}| \geq 2$) and let $\mathfrak{R}_{\mathcal{A}}$ and $\mathfrak{C}_{\mathcal{A}}$ be the set of finite and infinite continued fractions whose partial quotients belong to \mathcal{A} :

$$\begin{aligned}\mathfrak{R}_{\mathcal{A}} &= \{[d_1, \dots, d_k] : d_j \in \mathcal{A}, j = 1, \dots, k\}, \\ \mathfrak{C}_{\mathcal{A}} &= \{[d_1, \dots, d_j, \dots] : d_j \in \mathcal{A}, j = 1, \dots\}.\end{aligned}$$

And let

$$\mathfrak{D}_{\mathcal{A}}(N) = \left\{ d \mid d \leq N, \exists b : (b, d) = 1, \frac{b}{d} \in \mathfrak{R}_{\mathcal{A}} \right\}$$

be the set of denominators bounded by N . Let $\delta_{\mathcal{A}}$ be the Hausdorff dimension of $\mathfrak{C}_{\mathcal{A}}$. Then the Bourgain-Kontorovich's theorem [1, p. 13, Theorem 1.25] is as follows

Theorem 1.1. *For any alphabet \mathcal{A} with*

$$\delta_{\mathcal{A}} > 1 - \frac{5}{312} = 0,983914\dots, \quad (1.2)$$

the following inequality (positive proportion)

$$\#\mathfrak{D}_{\mathcal{A}}(N) \gg N. \quad (1.3)$$

holds.

For some alphabets the condition (1.2) can be verified by two means. For an alphabet $\mathcal{A} = \{1, 2, \dots, A-1, A\}$ Hensley [3] proved that

$$\delta_{\mathcal{A}} = \delta_A = 1 - \frac{6}{\pi^2} \frac{1}{A} - \frac{72 \log A}{\pi^4 A^2} + O\left(\frac{1}{A^2}\right). \quad (1.4)$$

Moreover Jenkinson [7] obtained approximate values for some $\delta_{\mathcal{A}}$. In view of these results the alphabet

$$\{1, 2, \dots, A-1, A\} \quad (1.5)$$

with $A = 50$ is assumed to satisfy (1.2). Several results improving (1.3) were also proved in [1]. However, we do not consider them in our work.

2 Statement of the main result

First of all we must state one lemma [1, p. 46, Lemma 7.1.]. The proof of our main result is essentially based on this lemma.

To begin with we describe all necessary objects. Let $K, X, Y \geq 1$ be real numbers and q be a positive integer. Moreover, let $\eta = (x, y)^t, \eta' = (u, v)^t \in \mathbb{Z}^2$ be vectors such as

$$|\eta| \asymp \frac{X}{Y}, |\eta'| \asymp X, (x, y) = 1, (u, v) = 1.$$

Lemma 2.1. (*[1, p. 46, Lemma 7.1.]*) If the following inequality

$$(qK)^{\frac{13}{5}} < Y < X, \quad (2.1)$$

holds, then

$$\#\left\{\gamma \in SL_2(\mathbb{Z}) \mid \|\gamma\| \asymp Y, |\gamma\eta - \eta'| < \frac{X}{K}, \gamma\eta \equiv \eta' \pmod{q}\right\} \ll \frac{Y^2}{(qK)^2}. \quad (2.2)$$

The main result of the paper is the following theorem.

Theorem 2.1. *For any alphabet \mathcal{A} with*

$$\delta_{\mathcal{A}} > 1 - \frac{27 - \sqrt{633}}{16} = 0,8849\dots, \quad (2.3)$$

the following inequality (positive proportion) holds

$$\#\mathfrak{D}_{\mathcal{A}}(N) \gg N. \quad (2.4)$$

Remark 2.1. *It is proved [7] that $\delta_7 = 0,8889\dots$. From this follows that the alphabet $\{1, 2, \dots, 7\}$ satisfies the condition of Theorem 2.1. It is also proved in [7] that for the alphabet $\mathcal{A} = \{1, 2, \dots, 6, 8\}$ one has $\delta_{\mathcal{A}} = 0,8851\dots$. Consequently, the alphabet $\mathcal{A} = \{1, 2, \dots, 6, 8\}$ also satisfies the condition of Theorem 2.1.*

Remark 2.2. *It seems to follow from the proof of the Lemma 2.1 that the condition (2.1) can be replaced by a weaker one*

$$(qK)^{\frac{64}{25}+\epsilon} < Y < X. \quad (2.5)$$

Then the statement of the Theorem 2.1 holds with

$$\delta_{\mathcal{A}} > 1 - \frac{25}{114 + 2\sqrt{2274}} = 0,8805\dots \quad (2.6)$$

Remark 2.3. *The proof of Lemma 2.1 given in the article [1] is based on the paper [2]. In this article using the spectral theory of automorphic forms statements similar to Lemma 2.1 are proved. In our next article we are going to prove a result similar to Lemma 2.1 using only estimates for Kloosterman sums. Certainly we are able to prove only a weaker result, however, it will suffice to prove that for the alphabet $\{1, 2, \dots, 12, 13\}$ the inequality (2.4) holds.*

Remark 2.4. *The proof of the Theorem 2.1 is based on the method of Bourgain-Kontorovich [1]. In this article a new technique for estimating exponential sums taking over a set of continuants is devised. We improve on this method by refining the main set Ω_N . In [1] this set is named as ensemble.*

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4 Notation

Throughout $\epsilon_0 = \epsilon_0(\mathcal{A}) \in (0, \frac{1}{2500})$. For two functions $f(x), g(x)$ the Vinogradov notation $f(x) \ll g(x)$ means that there exists a constant C , depending on A, ϵ_0 , such that $|f(x)| \leq Cg(x)$. The notation $f(x) = O(g(x))$ means the same. The notation $f(x) = O_1(g(x))$ means that $|f(x)| \leq g(x)$. The notation $f(x) \asymp g(x)$ means that $f(x) \ll g(x) \ll f(x)$. Also a traditional notation $e(x) = \exp(2\pi ix)$ is used. The cardinality of a finite set S is denoted either $|S|$ or $\#S$. $[\alpha]$ and $\|\alpha\|$ denote the integral part of α and the distance from α to the nearest integer: $[\alpha] = \max \{z \in \mathbb{Z} | z \leq \alpha\}$, $\|\alpha\| = \min \{|z - \alpha| | z \in \mathbb{Z}\}$. The following sum $\sum_{d|q} 1$ is denoted as $\tau(q)$.

Chapter II

Preparation for estimating exponential sums

5 Continuants and matrices

In this section we recall the simplest techniques concerning continuants. As a rule, all of them can be found in any research dealing with continued fractions. To begin with we define several operations on finite sequences. Let

$$D = \{d_1, d_2, \dots, d_k\}, B = \{b_1, b_2, \dots, b_n\} \quad (5.1)$$

then D, B denotes the following sequence

$$D, B = \{d_1, d_2, \dots, d_k, b_1, b_2, \dots, b_n\}.$$

For every D from (5.1) let define $D_-, D^-, \overleftarrow{D}$ as follows

$$D_- = \{d_2, d_3, \dots, d_k\}, D^- = \{d_1, d_2, \dots, d_{k-1}\}, \overleftarrow{D} = \{d_k, d_{k-1}, \dots, d_2, d_1\}.$$

We denote by $[D]$ the continued fraction (1.1), that is $[D] = [d_1, \dots, d_k]$. And by $\langle D \rangle$ we denote its denominator $\langle D \rangle = \langle d_1, \dots, d_k \rangle$. This denominator is called the continuant of the sequence D . The continuant of the sequence can also be defined as follows

$$\langle \rangle = 1, \langle d_1 \rangle = d_1,$$

$$\langle d_1, \dots, d_k \rangle = \langle d_1, \dots, d_{k-1} \rangle d_k + \langle d_1, \dots, d_{k-2} \rangle, \text{ для } k \geq 2.$$

It is well known [13] that

$$\langle D \rangle = \langle \overleftarrow{D} \rangle, [D] = \frac{\langle D_- \rangle}{\langle D \rangle}, [\overleftarrow{D}] = \frac{\langle D^- \rangle}{\langle D \rangle}, \quad (5.2)$$

$$\langle D, B \rangle = \langle D \rangle \langle B \rangle (1 + [\overleftarrow{D}][B]). \quad (5.3)$$

It follows from this that

$$\langle D \rangle \langle B \rangle \leq \langle D, B \rangle \leq 2 \langle D \rangle \langle B \rangle, \quad (5.4)$$

and that the elements of the matrix

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & d_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & d_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & d_k \end{pmatrix} \quad (5.5)$$

can be expressed by continuants

$$a = \langle d_2, d_3, \dots, d_{k-1} \rangle, b = \langle d_2, d_3, \dots, d_k \rangle, c = \langle d_1, d_2, \dots, d_{k-1} \rangle, d = \langle d_1, d_2, \dots, d_k \rangle. \quad (5.6)$$

For the matrix γ from (5.5) we use the following norm

$$\|\gamma\| = \max\{|a|, |b|, |c|, |d|\},$$

It follows from (5.6) that

$$\|\gamma\| = d = \langle d_1, d_2, \dots, d_k \rangle. \quad (5.7)$$

For γ from (5.5) we have $\det \gamma = (-1)^k$. So for an even k one has $\det \gamma = 1$, that is $\gamma \in SL(2, \mathbb{Z})$.

Let $\Gamma_{\mathcal{A}} \subseteq SL(2, \mathbb{Z})$ be a semigroup generated by

$$\begin{pmatrix} 0 & 1 \\ 1 & a_i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_j \end{pmatrix} = \begin{pmatrix} 1 & a_j \\ a_i & a_i a_j + 1 \end{pmatrix},$$

where $a_i, a_j \in \mathcal{A}$. It follows from (5.7) that to prove a positive density of continuants in \mathbb{N}

$$\#\mathfrak{D}_{\mathcal{A}}(N) \gg N, \quad (5.8)$$

it is enough to obtain the same property of the set

$$\|\Gamma_{\mathcal{A}}\| = \left\{ \|\gamma\| \mid \gamma \in \Gamma_{\mathcal{A}} \right\}. \quad (5.9)$$

In fact, for proving inequality (5.8) only a part of the semigroup $\Gamma_{\mathcal{A}}$, a so called ensemble Ω_N , will be used. A preparation for constructing Ω_N will start in the next section.

6 Hensley's method

Before estimating the amount of continuants not exceeding N it might be well to assess the amount of continued fractions with denominator being bounded by N . Though these problems are similar, in the second case every continuant should be counted in view of its multiplicity. Let $V_{\mathcal{A}}(k)$ be the set of words with the length k

$$V_{\mathcal{A}}(k) = \left\{ (d_1, d_2, \dots, d_k) \mid 1 \leq d_j \leq A, j = 1, \dots, k \right\},$$

and let $V_{\mathcal{A}} = \bigcup_{k \geq 1} V_{\mathcal{A}}(k)$ be the set of all finite words. Let

$$A = \max \mathcal{A}. \quad (6.1)$$

We will consider on alphabet \mathcal{A} only words having an even length. They can also be treated as words on the alphabet $(\mathcal{A}, \mathcal{A})$, that is, consisting of pairs (a, b) where $a, b \in \mathcal{A}$. Let us denote the alphabet $(\mathcal{A}, \mathcal{A})$ by \mathcal{A}^2 . Let $V_{\mathcal{A}^2}$ be the set of words on \mathcal{A} having an even length.

Let $\mathfrak{R}_{\mathcal{A}^2}$ be the set of finite continued fractions constructed from sequences in $V_{\mathcal{A}^2}$. We also write

$$\mathfrak{R}_{\mathcal{A}^2}(N) = \left\{ D \in V_{\mathcal{A}^2} \mid \langle D \rangle \leq N \right\},$$

$$F_{\mathcal{A}}(x) = \# \left\{ D \in V_{\mathcal{A}^2} \mid \langle D \rangle \leq x \right\} = \# \mathfrak{R}_{\mathcal{A}^2}(x).$$

And let $D_{\mathcal{A}^2}(N)$ be the set of denominators of fractions from $\mathfrak{R}_{\mathcal{A}^2}(N)$. Generalizing Hensley's method [4] one can prove that $2\delta_{\mathcal{A}^2} = 2\delta_{\mathcal{A}}$ and that the following theorem holds.

Theorem 6.1. *Let $\delta_{\mathcal{A}} > \frac{1}{2}$, then for any $x \geq 4A^2$ one has*

$$\frac{1}{32A^4} x^{2\delta_{\mathcal{A}}} \leq F_{\mathcal{A}}(x) - F_{\mathcal{A}}\left(\frac{x}{4A^2}\right) \leq F_{\mathcal{A}}(x) \leq 8x^{2\delta_{\mathcal{A}}}. \quad (6.2)$$

Hensley [4] proved this theorem for the alphabet \mathcal{A} of the form (1.5).

7 The basic ideas of the Bourgain-Kontorovich's method

In this section a notion of constructions necessary for proving Theorem 2.1 will be given.

In view of exponential sums, for studying the density of the set (5.9) it is natural to estimate the absolute value of the sum

$$S_N(\theta) := \sum_{\gamma \in \Omega_N} e(\theta \|\gamma\|) \quad (7.1)$$

where $\Omega_N \subseteq \Gamma_{\mathcal{A}} \cap \{\|\gamma\| \leq N\}$ is a proper set of matrices (ensemble), $\theta \in [0, 1]$, and the norm $\|\gamma\|$ is defined in (5.7). As usual, the Fourier coefficient of the function $S_N(\theta)$ is defined by

$$\widehat{S}_N(n) = \int_0^1 S_N(\theta) e(-n\theta) d\theta = \sum_{\gamma \in \Omega_N} \mathbf{1}_{\{\|\gamma\|=n\}}.$$

Note that if $\widehat{S}_N(n) > 0$ then $n \in \mathfrak{D}_{\mathcal{A}^2}(N)$. Since

$$S_N(0) = \sum_{\gamma \in \Omega_N} 1 = \sum_{n=1}^N \sum_{\gamma \in \Omega_N} \mathbf{1}_{\{\|\gamma\|=n\}} = \sum_{n=1}^N \widehat{S}_N(n) = \sum_{n=1}^N \widehat{S}_N(n) \mathbf{1}_{\{\widehat{S}_N(n) \neq 0\}},$$

then applying Cauchy-Schwarz inequality one has

$$(S_N(0))^2 \leq \sum_{n=1}^N \mathbf{1}_{\{\widehat{S}_N(n) \neq 0\}} \sum_{m=1}^N \left(\widehat{S}_N(m) \right)^2. \quad (7.2)$$

The first factor of the right hand side of the inequality (7.2) is bounded from above by $\#\mathfrak{D}_{\mathcal{A}^2}(N)$. Applying Parseval for the second factor one has

$$\sum_{n=1}^N \left(\widehat{S}_N(n) \right)^2 = \int_0^1 |S_N(\theta)|^2 d\theta.$$

Consequently

$$(S_N(0))^2 \leq \#\mathfrak{D}_{\mathcal{A}^2}(N) \int_0^1 |S_N(\theta)|^2 d\theta. \quad (7.3)$$

Thus a lower bound on the magnitude of the set $\mathfrak{D}_{\mathcal{A}^2}(N)$ follows from (7.3)

$$\#\mathfrak{D}_{\mathcal{A}^2}(N) \geq \frac{(S_N(0))^2}{\int_0^1 |S_N(\theta)|^2 d\theta}. \quad (7.4)$$

Thus, the estimate

$$\#\mathfrak{D}_{\mathcal{A}}(N) \gg N$$

will be proved, if we are able to assess exactly as possible the integral from (7.4)

$$\int_0^1 |S_N(\theta)|^2 d\theta \ll \frac{(S_N(0))^2}{N} = \frac{|\Omega_N|^2}{N}. \quad (7.5)$$

It follows from the Dirichlet's theorem that for any $\theta \in [0, 1]$ there exist $a, q \in \mathbb{N} \cup \{0\}$ and $\beta \in \mathbb{R}$ such that

$$\theta = \frac{a}{q} + \beta, \quad (a, q) = 1, \quad 0 \leq a \leq q \leq N^{1/2}, \quad \beta = \frac{K}{N}, \quad |K| \leq \frac{N^{1/2}}{q},$$

with $a = 0$ and $a = q$ being possible if only $q = 1$. Following the article [1], to obtain the estimate (7.5) we represent the integral as the sum of integrals over different domains in variables (q, K) . Each of them will be estimated in a special way depending on the domain.

It remains to define ensemble Ω_N . To begin with we determine a concept "pre-ensemble". The subset Ξ of matrices $\gamma \in \Gamma_{\mathcal{A}}$ is referred to as N – **pre-ensemble**, if the following conditions hold

1. for any matrix $\gamma \in \Xi$ its norm is of the order of N :

$$\|\gamma\| \asymp N; \quad (7.6)$$

2. for any $\epsilon > 0$ the set Ξ contains ϵ –full amount of elements, that is

$$\#\Xi \gg_{\epsilon} N^{2\delta_{\mathcal{A}} - \epsilon}. \quad (7.7)$$

By the **product** of two pre-ensembles $\Xi^{(1)}\Xi^{(2)}$ we mean the set of all possible products of matrices $\gamma_1\gamma_2$ such that $\gamma_1 \in \Xi^{(1)}$, $\gamma_2 \in \Xi^{(2)}$. The product of pre-ensembles has an **unique expansion** if it follows from the relations

$$\gamma_1\gamma_2 = \gamma'_1\gamma'_2, \quad \gamma_1, \gamma'_1 \in \Xi^{(1)}, \quad \gamma_2, \gamma'_2 \in \Xi^{(2)} \quad (7.8)$$

that

$$\gamma_1 = \gamma'_1, \quad \gamma_2 = \gamma'_2. \quad (7.9)$$

Let ϵ_0 be a fixed number such that $0 < \epsilon_0 \leq \frac{1}{2}$. Then N – pre-ensemble Ω is called the right (left) (ϵ_0, N) – **ensemble** if for any M , such that

$$1 \ll M \leq N^{\frac{1}{2}}, \quad (7.10)$$

there exist positive numbers N_1 and N_2 such that

$$N_1 N_2 \asymp N, \quad N_2^{1-\epsilon_0} \ll M \ll N_2, \quad (7.11)$$

and there exist N_1 –pre-ensemble $\Xi^{(1)}$ and N_2 –pre-ensemble $\Xi^{(2)}$ such that the pre-ensemble Ω is equal to the product $\Xi^{(1)}\Xi^{(2)}$ ($\Xi^{(2)}\Xi^{(1)}$ respectively) having an unique expansion. Such terminology allows us to say that in the article [1] the $(\frac{1}{2}, N)$ –ensemble has been constructed while we will construct (ϵ_0, N) –ensemble, being simultaneously the right and the left (bilateral) ensemble, for $\epsilon_0 \in (0, \frac{1}{2500})$.

Remark 7.1. *Observe that there is no use to require an upper bound in (7.7). According to the Theorem 6.1 it follows from (7.6) that*

$$\#\Xi \ll N^{2\delta_A}.$$

8 Pre-ensemble $\Xi(M)$.

Let $\delta := \delta_A > \frac{1}{2}$, $\Gamma := \Gamma_A$ and as usual $A = \max \mathcal{A}$. Let also M be a fixed parameter satisfying the inequality

$$M \geq 2^9 A^3 \log^3 M. \quad (8.1)$$

In this section we construct a pre-ensemble $\Xi(M) \subset \Gamma$. It is the key element which will be used to construct the ensemble Ω_N . To generate $\Xi(M)$ we use an algorithm. The number M is an input parameter. During the algorithm we generate the following numbers

$$L = L(M) \asymp M, \quad p = p(M) \asymp \log \log M, \quad k = k(M) \asymp \log M,$$

being responsible for the properties of the elements of $\Xi(M)$. We now proceed to the description of the algorithm consisting of four steps.

Step 1 First consider the set $S_1 \subset \Gamma$ of matrices $\gamma \in \Gamma$, such that

$$\frac{M}{64A^2} \leq \|\gamma\| \leq M. \quad (8.2)$$

According to the Theorem 6.1 $\#S_1 \asymp M^{2\delta}$.

Step 2 Let $F_0 = 0$, $F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$ be Fibonacci numbers. числа Фибоначи. Define an integer number $p = p(M)$ by the relation

$$F_{p-1} \leq \log^{\frac{1}{2}} M \leq F_p. \quad (8.3)$$

Note that then

$$F_p \leq 2F_{p-1} \leq 2 \log^{\frac{1}{2}} M. \quad (8.4)$$

Let consider the set $S_2 \subset S_1$ of matrices $\gamma \in S_1$ of the form (5.5) for witch

$$d_1 = d_2 = \dots = d_p = 1, \quad d_k = d_{k-1} = \dots = d_{k-p+1} = 1. \quad (8.5)$$

To put it another way, the first p and the last p elements of the sequence $D = D(\gamma) = \{d_1, d_2, \dots, d_k\}$ are equal to one. At the moment we have to interrupt for a while the description of the algorithm in order to prove the following lemma.

Lemma 8.1. *One has the estimate*

$$\#S_2 \geq \frac{M^{2\delta}}{2^{13} A^4 \log^2 M}. \quad (8.6)$$

Proof. The sequence D can be represented in the form Последовательность D представим как

$$D = \{\underbrace{1, 1, \dots, 1}_p, b_1, b_2, \dots, b_n, \underbrace{1, 1, \dots, 1}_p\}$$

then the required inequality (8.2) can be represented in the form

$$\frac{M}{64A^2} \leq \langle \underbrace{1, 1, \dots, 1}_p, b_1, b_2, \dots, b_n, \underbrace{1, 1, \dots, 1}_p \rangle \leq M. \quad (8.7)$$

Let prove that the inequality (8.7) follows from the inequality

$$\frac{M}{64A^2 \log M} \leq \langle b_1, b_2, \dots, b_n \rangle \leq \frac{M}{16 \log M}. \quad (8.8)$$

Indeed, let the inequality (8.8) be true. Then on the one side it follows from inequalities (5.4) and (8.4) that

$$\langle \underbrace{1, 1, \dots, 1}_p, b_1, b_2, \dots, b_n, \underbrace{1, 1, \dots, 1}_p \rangle \leq 4F_p^2 \langle b_1, b_2, \dots, b_n \rangle \leq 16 \langle b_1, b_2, \dots, b_n \rangle \log M \leq M,$$

and on the other side it follows in a similar way from the inequality (8.3) that

$$\langle \underbrace{1, 1, \dots, 1}_p, b_1, b_2, \dots, b_n, \underbrace{1, 1, \dots, 1}_p \rangle \geq F_p^2 \langle b_1, b_2, \dots, b_n \rangle \geq \langle b_1, b_2, \dots, b_n \rangle \log M \geq \frac{M}{64A^2}.$$

Thus the implication (8.8) \Rightarrow (8.7) is proved. It remains to obtain a lower bound for the amount of sequences $B = \langle b_1, b_2, \dots, b_n \rangle$ satisfying the inequality (8.8). We set $x = \frac{M}{16 \log M}$ and note that the condition $x \geq 4A^2$ in Theorem 6.1 follows from the inequality (8.1). Consequently, considering (6.2) one has

$$\#S_2 \geq F_{\mathcal{A}}(x) - F_{\mathcal{A}}\left(\frac{x}{4A^2}\right) \geq \frac{1}{32A^4} \left(\frac{M}{16 \log M}\right)^{2\delta} \geq \frac{M^{2\delta}}{2^{13} A^4 \log^2 M},$$

since $\delta < 1$. This completes the proof of the lemma. \square

Let return to the description of the algorithm.

Step 3 For any L in the interval

$$\left[\frac{M}{64A^2}, M \right] \quad (8.9)$$

consider the set $S_3(L) \subset S_2$ of matrices $\gamma \in S_2$, for which the following inequality holds

$$\max \left\{ \frac{M}{64A^2}, L(1 - \log^{-1} L) \right\} \leq \|\gamma\| \leq L. \quad (8.10)$$

Here, we also have to interrupt the description of the algorithm in order to prove the following lemma.

Lemma 8.2. *There is a number L in the interval (8.9) such that*

$$|S_3(L)| \geq \frac{L^{2\delta}}{2^{16} A^5 \log^3 L}. \quad (8.11)$$

Proof. Let t be the minimal positive integer number satisfying the inequality

$$(1 - \log^{-1} M)^t \leq \frac{1}{64A^2}. \quad (8.12)$$

Note that $t \leq 8A \log M$. For $j = 1, 2, \dots, t$ consider sets $s(j)$ each of them consists of matrices $\gamma \in S_2$, such that

$$M(1 - \log^{-1} M)^j \leq \|\gamma\| \leq M(1 - \log^{-1} M)^{j-1}$$

Since $S_2 \subset \bigcup_{1 \leq j \leq t} s(j)$, by the pigeonhole principle there is a set among $s(j)$ containing at least $\frac{|S_2|}{t}$ matrices. Let

$$L = M(1 - \log^{-1} M)^{j_0-1},$$

then L belongs to the segment (8.9) and $s(j_0) \subset S_3(L)$. Hence $|S_3(L)| \geq \frac{|S_2|}{t}$. Using the bound (8.6) and the restriction on t one has

$$|S_3(L)| \geq \frac{M^{2\delta}}{(2^{13} A^4 \log^2 M) 8A \log M} = \frac{M^{2\delta}}{2^{16} A^5 \log^3 M}. \quad (8.13)$$

Because the function $f(x) = x^{2\delta} \log^{-3} x$ increases and since $M \geq L$, then replacing in (8.13) the parameter M by L one has the inequality (8.11). This completes the proof of the lemma. \square

Returning to the algorithm we choose in the interval (8.9) any L (for example the maximal one) satisfying the inequality (8.11) and fix it. Now let $S_3 := S_3(L)$.

Step 4 For $\gamma \in S_3$ let $k(\gamma)$ be the length of the sequence $D(\gamma)$. Represent the set S_3 as the union of the sets $S_4(k)$, consisting of those matrices $\gamma \in S_3$ for which $k(\gamma) = k$ is fixed.

Lemma 8.3. *There exists k for which*

$$|S_4(k)| \geq \frac{L^{2\delta}}{2^{18} A^5 \log^4 L}. \quad (8.14)$$

Proof. Since for all $D \in V_A(r)$ the inequality $\langle D \rangle \geq \langle \underbrace{1, 1, \dots, 1}_r \rangle$, holds, then

$$\langle D \rangle \geq \left(\frac{\sqrt{5} + 1}{2} \right)^{r-1}$$

and consequently

$$k \leq \frac{\log \|\gamma\|}{\log \frac{\sqrt{5}+1}{2}} + 1 \leq 4 \log \|\gamma\| \leq 4 \log L. \quad (8.15)$$

Hence, by the pigeonhole principle, there is a k , for which

$$|S_4(k)| \geq \frac{|S_3|}{4 \log L} \geq \frac{L^{2\delta}}{(4 \log L) 2^{16} A^5 \log^3 L} = \frac{L^{2\delta}}{2^{18} A^5 \log^4 L}$$

by (8.11) and (8.15). This completes the proof of the lemma. \square

Returning to the algorithm we fix any k , satisfying the inequality (8.14) and write $S_4 := S_4(k)$. **Algorithm is completed.**

Now we write $\Xi(M) := S_4$. Recall the properties of matrices $\gamma \in \Xi(M)$. For any $\gamma \in \Xi(M)$ we have from the construction:

- i the first and the last p elements of the sequence $D(\gamma)$ are equal to 1, where p is defined by the inequality (8.3);
- ii $L(1 - \log^{-1} L) \leq \|\gamma\| \leq L$;
- iii $k(\gamma) = \text{const}$, that is, the length of $D(\gamma)$ is fixed for all $\gamma \in \Xi(M)$.

Besides, we have proved that

$$\#\Xi(M) \geq \frac{L^{2\delta}}{2^{18} A^5 \log^4 L}. \quad (8.16)$$

The first property allows us to prove an important lemma

Lemma 8.4. *For every matrix $\gamma \in \Xi(M)$ of the form $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the following inequalities hold*

$$\left| \frac{b}{d} - \frac{1}{\varphi} \right| \leq \frac{2}{\log L}, \quad \left| \frac{c}{d} - \frac{1}{\varphi} \right| \leq \frac{2}{\log L}, \quad (8.17)$$

where φ is the golden ratio

$$\varphi = 1 + [1, 1, \dots, 1, \dots] = \frac{\sqrt{5} + 1}{2}. \quad (8.18)$$

Proof. It follows from (5.2) and (5.6) that $\frac{b}{d} = [D(\gamma)]$, $\frac{c}{d} = [\overleftarrow{D(\gamma)}]$. Hence the fraction $\alpha = \underbrace{[1, 1, \dots, 1]}_p$ is a convergent fraction to $\frac{b}{d}$ and to $\frac{c}{d}$. The denominator of α is equal to F_p

and it follows from the choice of parameters (8.3), (8.9) that $F_p \geq \log^{\frac{1}{2}} L$. Hence,

$$\left| \frac{b}{d} - \alpha \right| \leq \frac{1}{\log L}, \quad \left| \frac{c}{d} - \alpha \right| \leq \frac{1}{\log L}. \quad (8.19)$$

But α is also a convergent fraction to $\frac{1}{\varphi}$, thus $\left| \alpha - \frac{1}{\varphi} \right| \leq \frac{1}{\log L}$. Applying the triangle inequality we obtain the desired inequalities. This completes the proof of the lemma. \square

9 Parameters and their properties

Let $N \geq N_{\min} = N_{\min}(\epsilon_0, \mathcal{A})$ and write

$$J = J(N) = \left\lceil \frac{\log \log N - 4 \log(10A) + 2 \log \epsilon_0}{-\log(1 - \epsilon_0)} \right\rceil, \quad (9.1)$$

where as usual $A \geq |\mathcal{A}| \geq 2$ and require the following inequality $J(N_{min}) \geq 10$ to hold. Using the definition (9.1), one has

$$\frac{10^4 A^4}{\log N} \leq \epsilon_0^2 (1 - \epsilon_0)^J \leq \frac{10^5 A^4}{\log N}. \quad (9.2)$$

Now let define a finite sequence

$$\{N_{-J-1}, N_{-J}, \dots, N_{-1}, N_0, N_1, \dots, N_{J+1}\}, \quad (9.3)$$

having set $N_{J+1} = N$ and

$$N_j = \begin{cases} N^{\frac{1}{2-\epsilon_0}(1-\epsilon_0)^{1-j}}, & \text{если } -1 - J \leq j \leq 1; \\ N^{1-\frac{1}{2-\epsilon_0}(1-\epsilon_0)^j}, & \text{если } 0 \leq j \leq J. \end{cases} \quad (9.4)$$

It is obvious that the sequence is well-defined for $j = 0$ and $j = 1$.

Lemma 9.1. 1. For $-J \leq m \leq J - 1$ the following equation holds

$$N_{-m} N_{m+1} = N. \quad (9.5)$$

2. For $-J - 1 \leq m \leq J - 1$ the following relations hold

$$\frac{N_{m+1}}{N_m} = \begin{cases} N_{m+1}^{\epsilon_0}, & \text{if } m \leq 0; \\ \left(\frac{N}{N_m}\right)^{\epsilon_0}, & \text{if } m \geq 0, \end{cases} \quad (9.6)$$

$$\frac{N_{m+1}}{N_m} = N^{\frac{\epsilon_0}{2-\epsilon_0}(1-\epsilon_0)^{|m|}}, \quad (9.7)$$

$$N_m \geq N_{m+1}^{1-\epsilon_0}. \quad (9.8)$$

3. Для $-1 \leq j < h \leq J + 1$ выполнено

$$N_{h-J}^{(1-\epsilon_0)^{h-j}} = N_{j-J}. \quad (9.9)$$

Proof. All propositions follow directly from the definition (9.4). This completes the proof of the lemma. \square

Lemma 9.2. For $-J \leq m \leq J - 1$ the following estimate holds

$$\frac{N_{m+1}}{N_m} \geq \exp\left(\frac{10^4 A^4}{2\epsilon_0}\right); \quad (9.10)$$

moreover

$$\exp\left(\frac{10^4 A^4}{2\epsilon_0^2}\right) \leq \frac{N}{N_J} = N_{1-J} \leq \exp\left(\frac{10^5 A^4}{\epsilon_0^2}\right). \quad (9.11)$$

Proof. The inequality (9.10) follows from (9.7) and the lower bound in (9.2). Now let prove the inequality (9.11). The equation $\frac{N}{N_J} = N_{1-J}$ follows from (9.5) with $m = -J$. The estimate of N_{1-J} follows from (9.4) and (9.2). This completes the proof of the lemma. \square

Lemma 9.3. *For any M , such that*

$$N_{1-J} \leq M \leq N_J, \quad (9.12)$$

there exist indexes j and h , such that

$$2 \leq j \leq 2J, \quad 1 \leq h \leq 2J-1, \quad h = 2J - j + 1, \quad (9.13)$$

for which the following inequalities hold

$$N_{j-J}^{1-\epsilon_0} \leq M \leq N_{j-J}, \quad (9.14)$$

$$\left(\frac{N}{N_{h-J}} \right)^{1-\epsilon_0} \leq M \leq \frac{N}{N_{h-J}}. \quad (9.15)$$

Proof. Since the sequence $\{N_j\}$ is increasing there exists the index j in (9.13) such that

$$N_{j-1-J} \leq M \leq N_{j-J}, \quad (9.16)$$

then (9.14) follows from (9.8). The inequality (9.15) can be obtained by substituting the equation (9.5) into (9.14). This completes the proof of the lemma. \square

For a nonnegative integer number n we write

$$\tilde{N}_{n-J} = \begin{cases} N_{n-J}, & \text{if } n \geq 1; \\ 1, & \text{if } n = 0. \end{cases} \quad (9.17)$$

Moreover for integers j and h such that

$$0 \leq j < h \leq 2J+1, \quad (9.18)$$

we write

$$j_0(j, h) = \min \left\{ |n - J - 1| \mid j+1 \leq n \leq h \right\}. \quad (9.19)$$

Note that there are only three alternatives for the value of j_0

$$j_0 \in \{j - J, 0, J + 1 - h\}. \quad (9.20)$$

Lemma 9.4. *For integers j and h from (9.18) the following estimate holds*

$$\prod_{n=j+1}^h \left(2^9 A^3 \log \frac{N_{n-J}}{\tilde{N}_{n-1-J}} \right) \leq ((1 - \epsilon_0)^{j_0} \log N)^{\frac{7}{4}(h-j)}, \quad (9.21)$$

where $j_0 = j_0(j, h)$.

Proof. Consider two cases depending on the value of j .

1) $j > 0$. Using (9.10) if $n < 2J + 1$ and (9.11) if $n = 2J + 1$, for any n in the segment $j + 1 \leq n \leq h$ we obtain

$$2^9 A^3 \leq \frac{2^9}{10^3} (2\epsilon_0)^{3/4} \left(\log \frac{N_{n-J}}{N_{n-1-J}} \right)^{3/4} \leq \left(\log \frac{N_{n-J}}{N_{n-1-J}} \right)^{3/4}. \quad (9.22)$$

Hence, since for $j > 0$ one has $\tilde{N}_{n-1-J} = N_{n-1-J}$, we obtain

$$\prod_{n=j+1}^h \left(2^9 A^3 \log \frac{N_{n-J}}{\tilde{N}_{n-1-J}} \right) \leq \prod_{n=j+1}^h \left(\log \frac{N_{n-J}}{N_{n-1-J}} \right)^{7/4}. \quad (9.23)$$

Making allowance for $\epsilon_0 \in (0, \frac{1}{2500})$, it follows from (9.7) if $n < 2J + 1$ and from (9.4) if $n = 2J + 1$, that

$$\log \frac{N_{n-J}}{N_{n-1-J}} \leq \frac{1}{2 - \epsilon_0} (1 - \epsilon_0)^{|n-J-1|} \log N \leq (1 - \epsilon_0)^{j_0} \log N \quad (9.24)$$

Substituting the estimate (9.24) into (9.23) we obtain the statement of the theorem in case $j > 0$.

2) $j = 0$. Using the lower bound from (9.11) we obtain

$$2^9 A^3 \leq \frac{2^9}{10^3} (2\epsilon_0^2)^{3/4} (\log N_{1-J})^{3/4} \leq (\log N_{1-J})^{3/4}. \quad (9.25)$$

It follows from the definition (9.17) and the result of the previous item that

$$\begin{aligned} \prod_{n=j+1}^h \left(2^9 A^3 \log \frac{N_{n-J}}{\tilde{N}_{n-1-J}} \right) &= 2^9 A^3 \log N_{1-J} \prod_{n=2}^h \left(2^9 A^3 \log \frac{N_{n-J}}{\tilde{N}_{n-1-J}} \right) \leq \\ &\leq (\log N_{1-J})^{7/4} ((1 - \epsilon_0)^{j_0(1,h)} \log N)^{\frac{7}{4}(h-j-1)}. \end{aligned} \quad (9.26)$$

We obtain by the definition (9.4) that

$$\log N_{1-J} = \frac{1}{2 - \epsilon_0} (1 - \epsilon_0)^J \log N \leq (1 - \epsilon_0)^J \log N.$$

Substituting this estimate into (9.26) we obtain

$$\prod_{n=j+1}^h \left(2^9 A^3 \log \frac{N_{n-J}}{\tilde{N}_{n-1-J}} \right) \leq ((1 - \epsilon_0)^{j_0(1,h)} \log N)^{\frac{7}{4}(h-j)} (1 - \epsilon_0)^{\frac{7}{4}(J-j_0(1,h))}. \quad (9.27)$$

Taking account of $j_0 \leq J$ and $1 - \epsilon_0 < 1$, we obtain

$$\prod_{n=j+1}^h \left(2^9 A^3 \log \frac{N_{n-J}}{\tilde{N}_{n-1-J}} \right) \leq ((1 - \epsilon_0)^{j_0(1,h)} \log N)^{\frac{7}{4}(h-j)}. \quad (9.28)$$

The fact that $j_0(1, h) = j_0(0, h)$ completes the proof.

Lemma is proved. □

10 The ensemble: constructing the set Ω_N .

In this section we construct a set Ω_N . It will be proved in §11 that this set is (ϵ_0, N) – ensemble. We construct the set by the inductive algorithm with the steps numbered by indexes $1, 2, \dots, 2J + 1$.

1. The first (the starting) step.

We set

$$M = M_1 = N_{1-J}. \quad (10.1)$$

Because of the lower bound in (9.11) the condition (8.1) obviously holds. So we can run the algorithm of §8 to generate the set $\Xi(M)$. During the algorithm we also obtain the numbers $L = L(M), p = p(M), k = k(M)$. By the construction the number L belongs to the segment $\left[\frac{M}{64A^2}, M\right]$, so we can set

$$L = \alpha_1 M_1 = \alpha_1 N_{1-J}, \quad (10.2)$$

where α_1 is a number from

$$\left[\frac{1}{64A^2}, 1\right]. \quad (10.3)$$

Let rename the returned pre-ensemble $\Xi(M)$ and numbers L, p and k to

$$\Xi(M) = \Xi_1, L = L_1, p = p_1, k = k_1.$$

For the next step of the algorithm we define the number M_2

$$M_2 = \frac{N_{2-J}}{(1 + \varphi^{-2})\alpha_1 N_{1-J}}, \quad (10.4)$$

where φ id from (8.18).

2. The step with the number j , where $2 \leq j \leq 2J + 1$ (the inductive step).

Write $M = M_j$. According to the inductive assumption the number M_j has been defined on the previous step by the formula

$$M_j = \frac{N_{j-J}}{(1 + \varphi^{-2})\alpha_{j-1} N_{j-1-J}}, \quad (10.5)$$

where α_{j-1} is a number from (10.3). To verify for such M the condition (8.1) it is sufficient to apply bounds of Lemma 9.2 having put $m = j - J$. Hence we can run the algorithm of §8 to generate $\Xi(M)$. Besides there exists a number α_j from the interval (10.3) such that for the parameter L the following equation holds

$$L = \alpha_j M = \frac{\alpha_j N_{j-J}}{(1 + \varphi^{-2})\alpha_{j-1} N_{j-1-J}}. \quad (10.6)$$

We rename $\Xi(M)$ to Ξ_j , the number L to L_j , the quantity p to p_j , the length k to k_j . If $j \leq 2J$, then the number M_{j+1} , which will be used in the next step, is defined by the equation

$$M_{j+1} = \frac{N_{j+1-J}}{(1 + \varphi^{-2})\alpha_j N_{j-J}}.$$

If $j = 2J + 1$, then the notation M_{j+1} is of no use, as the algorithm is completed.

We now define the ensemble Ω_N writing all the sets generated in the algorithm for one another

$$\Omega_N = \Xi_1 \Xi_2 \dots \Xi_{2J} \Xi_{2J+1}.$$

It means that the set Ω_N consists of all possible products of the form

$$\gamma_1 \gamma_2 \dots \gamma_{2J} \gamma_{2J+1}, \text{ with } \gamma_1 \in \Xi_1, \gamma_2 \in \Xi_2, \dots, \gamma_{2J+1} \in \Xi_{2J+1}.$$

To prove that Ω_N is really an ensemble we need two technical lemmas concerning quantities L_j . We will use the following notation

$$f(x) = O_1(g(x)), \text{ if } |f(x)| \leq g(x). \quad (10.7)$$

Lemma 10.1. *The following inequality holds*

$$\sum_{n=1}^{2J+1} \frac{1}{\log L_n} \leq \frac{1}{16000}. \quad (10.8)$$

Proof. Let prove that numbers L_j satisfy the following inequality

$$L_j \geq \frac{1}{64A^2(1+\varphi^{-2})} \frac{N_{j-J}}{N_{j-J-1}} \geq \frac{N_{j-J}}{100A^2 N_{j-J-1}}. \quad (10.9)$$

Actually, for $j > 1$ the inequality (10.9) follows from the definition (10.6). For $j = 1$ to deduce the same inequality (10.9) from (10.2) it is sufficient to know that $N_{-J} \geq 1$. The last inequality follows from inequalities (9.8) and (9.11) with $m = -J$:

$$N_{-J} \geq N_{1-J}^{1-\epsilon_0} \geq \exp\left(\frac{10^4 A^4}{2\epsilon_0^2}(1-\epsilon_0)\right) \geq 1.$$

Thus, the inequality (10.9) is proved. It follows from the bound (9.10) that

$$\left(\frac{N_m}{N_{m-1}}\right)^{1/2} \geq \exp\left(\frac{10^4 A^4}{4}\right) > 100A^2, \quad -J \leq m \leq J-1.$$

In that case, if $j \leq 2J$, then using (9.7) the estimate (10.9) can be resumed

$$L_j \geq \left(\frac{N_{j-J}}{N_{j-J-1}}\right)^{\frac{1}{2}} \geq N^{\frac{1}{4}\epsilon_0(1-\epsilon_0)^{|j-J-1|}}, \quad (10.10)$$

Hence

$$\log L_j \geq \frac{1}{4}\epsilon_0(1-\epsilon_0)^{|j-J-1|} \log N, \quad j \leq 2J. \quad (10.11)$$

If $j = 2J+1$, then from the lower bound in (9.11) we obtain in a similar way

$$L_{2J+1} \geq \left(\frac{N_{J+1}}{N_J}\right)^{\frac{1}{2}} \geq \exp\left(\frac{10^4 A^4}{4\epsilon_0^2}\right),$$

whence it follows that

$$\log L_j \geq \frac{10^4 A^4}{4\epsilon_0^2}. \quad (10.12)$$

Substituting the estimates (10.11) and (10.12) into the sum in (10.8) one has

$$\sum_{n=1}^{2J+1} \frac{1}{\log L_n} \leq \frac{4}{\epsilon_0} \sum_{n=0}^{2J} \frac{1}{(1-\epsilon_0)^{|n-J|} \log N} + \frac{4\epsilon_0^2}{10^4 A^4} \leq \frac{8}{\epsilon_0} \sum_{n=0}^J \frac{1}{(1-\epsilon_0)^n \log N} + \frac{4\epsilon_0^2}{10^4 A^4}. \quad (10.13)$$

Let estimate the geometric progression from (10.13):

$$\sum_{n=0}^J \frac{1}{\epsilon_0(1-\epsilon_0)^n} \leq \frac{1}{\epsilon_0(1-\epsilon_0)^J} \sum_{n=0}^{\infty} (1-\epsilon_0)^n \leq \frac{1}{\epsilon_0^2(1-\epsilon_0)^J} \leq \frac{\log N}{10^4 A^4},$$

since (9.2). Substituting this bound into (10.13) we obtain

$$\sum_{n=1}^{2J+1} \frac{1}{\log L_n} \leq \frac{8}{\log N} \frac{\log N}{10^4 A^4} + \frac{4\epsilon_0^2}{10^4 A^4} = \frac{4(2+\epsilon_0^2)}{10^4 A^4} < \frac{1}{10^3 A^4} \leq \frac{1}{16000}.$$

This completes the proof of the lemma. \square

To state the following lemma we suppose the real numbers

$$\Pi_1, \Pi_2, \dots, \Pi_{2J+1}$$

to satisfy the relations

$$\Pi_j = (1 + 2O_1(\log^{-1} L_j))^2 \prod_{n=1}^{j-1} (1 + 2O_1(\log^{-1} L_n))^3, \quad (10.14)$$

where the product over the empty set is regarded to be equal to one.

Lemma 10.2. *For any $j = 1, 2, \dots, 2J+1$ the following bound holds*

$$\exp(-10^{-3}) \leq \Pi_j \leq \exp(10^{-3}). \quad (10.15)$$

Proof. Taking the logarithm of the equation (10.14) and bounding from above the absolute value of the sum by the sum of absolute values we obtain

$$\begin{aligned} |\log \Pi_j| &\leq 2|\log(1 + 2O_1(\log^{-1} L_j))| + 3 \sum_{n=1}^{j-1} |\log(1 + 2O_1(\log^{-1} L_n))| \leq \\ &\leq 3 \sum_{n=1}^{2J+1} |\log(1 + 2O_1(\log^{-1} L_n))|. \end{aligned} \quad (10.16)$$

In view of Lemma 10.1, every number $\log^{-1} L_n$ for $n = 1, 2, \dots, 2J+1$ is less than $\frac{1}{16000}$; and in particular every number $2O_1(\log^{-1} L_n)$ belongs to the segment $[-\frac{1}{2}, \frac{1}{2}]$. But for any z in the segment $-\frac{1}{2} \leq z \leq \frac{1}{2}$ the inequality $|\log(1+z)| \leq |z| \log 4$ holds. Then by (10.16) we obtain

$$|\log \Pi_j| \leq 3 \sum_{n=1}^{2J+1} |2O_1(\log^{-1} L_n)| \log 4 < 12 \sum_{n=1}^{2J+1} \log^{-1} L_n. \quad (10.17)$$

Using Lemma 10.1 we obtain

$$|\log \Pi_j| \leq \frac{12}{16000} < 10^{-3}. \quad (10.18)$$

The inequality (10.15) follows immediately from (10.18). This completes the proof of the lemma. \square

11 Properties of Ω_N . It is really an ensemble!

In this section we prove that the constructed set Ω_N is an ensemble, that is, it satisfies the definition of ensemble in §7. Unique expansion is the easiest property to verify. Actually, if

$$\begin{aligned}\Omega^{(1)} &= \Xi_1 \Xi_2 \dots \Xi_j, \\ \Omega^{(2)} &= \Xi_{j+1} \Xi_{j+2} \dots \Xi_{2J+1},\end{aligned}$$

then firstly $\Omega_N = \Omega^{(1)} \Omega^{(2)}$. Secondly, as the representation of a matrix in the form (5.5) is unique then the implication (7.8) \Rightarrow (7.9) holds since the length $D(\gamma) = k_j$ is fixed for all $\gamma \in \Xi_j$ (the property (iii) in §7), for each $j = 1, 2, \dots, 2J + 1$.

The next purpose is to prove that Ω_N is a pre-ensemble.

Lemma 11.1. *For any j in the segment*

$$1 \leq j \leq 2J + 1, \quad (11.1)$$

for any collection of matrices

$$\xi_1 \in \Xi_1, \xi_2 \in \Xi_2, \dots, \xi_j \in \Xi_j, \quad (11.2)$$

one can find a number Π_j , satisfying the equality (10.14), such that

$$\|\xi_1 \xi_2 \dots \xi_j\| = \alpha_j N_{j-J} \Pi_j. \quad (11.3)$$

Proof. Let first $j = 1$. Then, by the construction of the pre-ensemble Ξ_1 (§8) and by the equation (10.2), the following equation holds

$$\|\xi_1\| = \alpha_1 N_{1-J} (1 + O_1(\log^{-1} L_1)). \quad (11.4)$$

Since

$$1 + O_1(\log^{-1} L_1) = (1 + 2O_1(\log^{-1} L_1))^2 = \Pi_1,$$

then substituting the last equation into (11.4) one has

$$\|\xi_1\| = \alpha_1 N_{1-J} \Pi_1, \quad (11.5)$$

and in the case $j = 1$ lemma is proved.

We now assume that lemma is proved for some j , such that $1 \leq j \leq 2J$, and prove that it holds for $j + 1$. It follows from (5.3) that

$$\|\xi_1 \xi_2 \dots \xi_j \xi_{j+1}\| = \|\xi_1 \xi_2 \dots \xi_j\| \|\xi_{j+1}\| (1 + [\overleftarrow{D}(\xi_j), \overleftarrow{D}(\xi_{j-1}), \dots, \overleftarrow{D}(\xi_1)][D(\xi_{j+1})]), \quad (11.6)$$

where $D(\gamma)$, as usual, denotes the sequence $D(\gamma) = \{d_1, d_2, \dots, d_k\}$, where

$$\gamma = \begin{pmatrix} 0 & 1 \\ 1 & d_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & d_2 \end{pmatrix} \dots \begin{pmatrix} 0 & 1 \\ 1 & d_k \end{pmatrix}.$$

It follows immediately from Lemma 8.4 that

$$[D(\xi_{j+1})] = \varphi^{-1} + 2O_1(\log^{-1} L_{j+1}), [\overleftarrow{D}(\xi_j) \dots, \overleftarrow{D}(\xi_1)] = \varphi^{-1} + 2O_1(\log^{-1} L_j). \quad (11.7)$$

Substituting (11.7) into (11.6), we obtain

$$\|\xi_1 \xi_2 \dots \xi_j \xi_{j+1}\| = \|\xi_1 \xi_2 \dots \xi_j\| \|\xi_{j+1}\| (1 + \varphi^{-2}) (1 + 2O_1(\log^{-1} L_j)) (1 + 2O_1(\log^{-1} L_{j+1})). \quad (11.8)$$

By the inductive hypothesis we have

$$\|\xi_1 \xi_2 \dots \xi_j\| = \alpha_j N_{j-J} \Pi_j. \quad (11.9)$$

And by the construction of the ensemble Ω_N (§10, "The step with the number $j + 1$.") the following equation holds

$$\|\xi_{j+1}\| = L_{j+1} (1 + O_1(\log^{-1} L_{j+1})) = \frac{\alpha_{j+1} N_{j+1-J}}{(1 + \varphi^{-2}) \alpha_j N_{j-J}} (1 + O_1(\log^{-1} L_{j+1})). \quad (11.10)$$

Substituting (11.9) and (11.10) into (11.8) and making cancelations, we obtain

$$\|\xi_1 \xi_2 \dots \xi_j \xi_{j+1}\| = \alpha_{j+1} N_{j+1-J} \tilde{\Pi}_{j+1}, \quad (11.11)$$

where

$$\tilde{\Pi}_{j+1} = (1 + O_1(\log^{-1} L_{j+1})) (1 + 2O_1(\log^{-1} L_j)) (1 + 2O_1(\log^{-1} L_{j+1})) \Pi_j. \quad (11.12)$$

Using the definition of Π_j by the equation (10.14) we obtain

$$\tilde{\Pi}_{j+1} = (1 + 2O_1(\log^{-1} L_{j+1}))^2 (1 + 2O_1(\log^{-1} L_j)) \Pi_j = \Pi_{j+1}$$

and hence

$$\|\xi_1 \xi_2 \dots \xi_j \xi_{j+1}\| = \alpha_{j+1} N_{j+1-J} \Pi_{j+1}.$$

The lemma is proved. \square

Lemma 11.2. *For any collection of matrices (11.2), for any numbers j, h in the interval*

$$1 \leq j \leq 2J + 1, \quad j < h \leq 2J + 1 \quad (11.13)$$

the following inequalities hold

$$\frac{1}{70A^2} N_{j-J} \leq \|\xi_1 \xi_2 \dots \xi_j\| \leq 1, 01 N_{j-J}, \quad (11.14)$$

$$\frac{1}{70A^2} N \leq \|\xi_1 \xi_2 \dots \xi_{2J+1}\| \leq 1, 01 N, \quad (11.15)$$

$$\frac{1}{150A^2} \frac{N_{h-J}}{N_{j-J}} \leq \|\xi_{j+1} \xi_{j+2} \dots \xi_h\| \leq 73A^2 \frac{N_{h-J}}{N_{j-J}}; \quad (11.16)$$

moreover, for $j \leq 2J$ one has

$$\frac{1}{150A^2} \frac{N}{N_{j-J}} \leq \|\xi_{j+1} \xi_{j+2} \dots \xi_{2J+1}\| \leq 73A^2 \frac{N}{N_{j-J}}. \quad (11.17)$$

Proof. First we prove the inequality (11.14). Recall that by the construction of the set Ω_N the following inequality holds

$$\frac{1}{64A^2} \leq \alpha_j \leq 1, \quad (11.18)$$

and by Lemma 10.2 we also have

$$\exp(-10^{-3}) \leq \Pi_j \leq \exp(10^{-3}). \quad (11.19)$$

Substituting (11.18) and (11.19) into (11.3), we obtain (11.14). In particular, as $N_{J+1} = N$, then by using (11.14) for $j = 2J + 1$ we obtain (11.15).

Now we prove the inequality (11.16). To do this we denote

$$W(j, h) = \|\xi_j \xi_{j+1} \dots \xi_h\|$$

and rewrite the inequality (5.4) in the form

$$W(1, j)W(j + 1, h) \leq W(1, h) \leq 2W(1, j)W(j + 1, h).$$

Hence, applying the inequality (11.14) twice we obtain

$$W(j + 1, h) \geq \frac{W(1, h)}{2W(1, j)} \geq \frac{N_{h-J}/(70A^2)}{2,02N_{j-J}} \geq \frac{1}{150A^2} \frac{N_{h-J}}{N_{j-J}},$$

and in the same way

$$W(j + 1, h) \leq \frac{W(1, h)}{W(1, j)} \leq \frac{1,01N_{h-J}}{N_{j-J}/(70A^2)} \leq 73A^2 \frac{N_{h-J}}{N_{j-J}}.$$

These prove the inequality (11.16). Putting $h = 2J + 1$ in it we obtain (11.17). The lemma is proved. \square

For integers j and h , such that

$$0 \leq j < h \leq 2J + 1, \quad (11.20)$$

we put

$$\Omega(j, h) = \Xi_{j+1} \Xi_{j+2} \dots \Xi_h. \quad (11.21)$$

Lemma 11.3. *The following estimate holds*

$$|\Omega(0, j)| \leq 9N_{j-J}^{2\delta}. \quad (11.22)$$

Proof. By definition, for $\gamma \in \Omega(0, j)$ one has $\gamma = \xi_1 \xi_2 \dots \xi_j$ for a collection of matrices (11.2). So, it follows from the inequality (11.14) that

$$\|\gamma\| \leq 1,01N_{j-J}. \quad (11.23)$$

The number of matrices γ , satisfying the inequality (11.23) can be bounded by Theorem 6.1. Estimating the result from above we obtain (11.22). This completes the proof of the lemma. \square

Recall that parameters \tilde{N}_{n-J} and $j_0(j, h)$ were introduced by formulae (9.17) and (9.19). Note that the restrictions (11.20) on j and h coincide with the restrictions (9.18).

Lemma 11.4. *For j and h in (11.20) the following bound holds*

$$|\Omega(j, h)| \geq \frac{1}{((1 - \epsilon_0)^{j_0} \log N)^{7(h-j)}} \left(\frac{N_{h-J}}{\tilde{N}_{j-J}} \right)^{2\delta}, \quad (11.24)$$

where $j_0 = j_0(j, h)$.

Proof. Multiplying the lower bounds (8.16) we obtain

$$|\Omega(j, h)| \geq \prod_{n=j+1}^h |\Xi_n| \geq \prod_{n=j+1}^h \frac{L_n^{2\delta}}{2^{18} A^5 \log^4 L_n}. \quad (11.25)$$

It follows from formulae (10.2) and (10.6) that

$$\frac{N_{n-J}}{c_1 \tilde{N}_{n-1-J}} \leq L_n \leq \frac{c_1 N_{n-J}}{\tilde{N}_{n-1-J}}, \quad (11.26)$$

where

$$c_1 = 64A^2(1 + \varphi^{-2}) \leq 2^7 A^2. \quad (11.27)$$

Let estimate the product of the numerators in (11.25). Applying (11.26) and (11.27) we obtain

$$\prod_{n=j+1}^h L_n^{2\delta} \geq \prod_{n=j+1}^h \left(\frac{N_{n-J}}{2^7 A^2 \tilde{N}_{n-1-J}} \right)^{2\delta}$$

After the cancelations we obtain

$$\prod_{n=j+1}^h L_n^{2\delta} \geq \left(\frac{N_{h-J}}{\tilde{N}_{j-J}} \right)^{2\delta} \prod_{n=j+1}^h (2^7 A^2)^{-2\delta}. \quad (11.28)$$

So the estimate (11.25) can be resumed in such a way

$$\begin{aligned} |\Omega(j, h)| &\geq \left(\frac{N_{h-J}}{\tilde{N}_{j-J}} \right)^{2\delta} \prod_{n=j+1}^h \frac{(2^7 A^2)^{-2\delta}}{2^{18} A^5 \log^4 L_n} \geq \\ &\geq \left(\frac{N_{h-J}}{\tilde{N}_{j-J}} \right)^{2\delta} \left(\prod_{n=j+1}^h (2^8 A^3 \log L_n) \right)^{-4}. \end{aligned} \quad (11.29)$$

The last product in (11.29) will be estimated separately. In view of the upper bound in (11.26) we have

$$\prod_{n=j+1}^h (2^8 A^3 \log L_n) \leq \prod_{n=j+1}^h \left(2^8 A^3 \left(\log(2^7 A^2) + \log \frac{N_{n-J}}{\tilde{N}_{n-1-J}} \right) \right). \quad (11.30)$$

Applying Lemma 9.2, we obtain

$$\log \frac{N_{n-J}}{\widetilde{N}_{n-1-J}} \geq \frac{10^4 A^4}{2\epsilon_0} \geq \log(2^7 A^2), \quad (11.31)$$

hence, applying Lemma 9.4, we obtain

$$\prod_{n=j+1}^h (2^8 A^3 \log L_n) \leq \prod_{n=j+1}^h \left(2^9 A^3 \log \frac{N_{n-J}}{\widetilde{N}_{n-1-J}} \right) \leq ((1 - \epsilon_0)^{j_0} \log N)^{\frac{7}{4}(h-j)}. \quad (11.32)$$

Substituting the estimate (11.32) into (11.29), we obtain (11.24). This completes the proof of the lemma. \square

Theorem 11.1. *For j and h in (11.20) the following estimate holds*

$$|\Omega(j, h)| \geq \left(\frac{N_{h-J}}{\widetilde{N}_{j-J}} \right)^{2\delta} \exp \left(- \left(\frac{\log \log N}{\log(1 - \epsilon_0)} + j_0 \right)^2 \right), \quad (11.33)$$

where $j_0 = j_0(j, h)$.

Proof. It follows from (11.24) that it is enough to prove the inequality

$$\exp \left(\left(\frac{\log \log N}{\log(1 - \epsilon_0)} + j_0 \right)^2 \right) \geq \exp (7(h - j) (\log \log N + j_0 \log(1 - \epsilon_0))).$$

Hence, it is sufficient to prove

$$7(h - j) (\log \log N + j_0 \log(1 - \epsilon_0)) \leq \left(\frac{\log \log N + j_0 \log(1 - \epsilon_0)}{\log(1 - \epsilon_0)} \right)^2.$$

One can readily obtain from (9.1) that

$$J \leq \frac{\log \log N}{-\log(1 - \epsilon_0)} - 1, \quad (11.34)$$

and since $j_0 \leq J$, so one has

$$\log \log N + j_0 \log(1 - \epsilon_0) > 0.$$

Thus, it is sufficient to prove that

$$7(h - j) \leq \frac{\log \log N + j_0 \log(1 - \epsilon_0)}{\log^2(1 - \epsilon_0)} = \frac{1}{-\log(1 - \epsilon_0)} \left(\frac{\log \log N}{-\log(1 - \epsilon_0)} - j_0 \right). \quad (11.35)$$

We observe that as $\epsilon_0 \in (0, \frac{1}{2500})$, so

$$-7 \log(1 - \epsilon_0) \leq \frac{1}{7}.$$

It follows from this, (11.34) and (11.35) that it is sufficient to prove

$$\frac{1}{7}(h - j) \leq J + 1 - j_0. \quad (11.36)$$

For $j_0 = 0$ the inequality (11.36) follows from the trivial bound $h - j \leq 2J + 1$. For $j_0 \neq 0$ (hence, $j > J$ or $h \leq J$) it follows from (9.19) and (9.20) that

$$h - j \leq \begin{cases} 2J + 1 - j = J + 1 - j_0, & \text{если } j > J; \\ h = J + 1 - j_0, & \text{если } h \leq J. \end{cases} \quad (11.37)$$

Hence the inequality (11.36) follows. This completes the proof of the theorem. \square

Corollary 11.1. *For $N \geq \exp(\epsilon_0^{-5})$ the following estimate holds*

$$N^{2\delta - \epsilon_0} \leq N^{2\delta} \exp\left(-\left(\frac{\log \log N}{\epsilon_0}\right)^2\right) \leq \#\Omega_N \leq 9N^{2\delta}. \quad (11.38)$$

Proof. To prove the upper bound we apply Lemma 11.3 with $j = 0$, $h = 2J + 1$. To prove the lower bound we use Theorem 11.1 putting $j = 0$, $h = 2J + 1$ in it (hence, $j_0 = 0$.) As $\epsilon_0 \in (0, \frac{1}{2500})$, so we have $\log^2(1 - \epsilon_0) > \epsilon_0^2$ and obtain the lower bound in (11.38). The corollary is proved. \square

Corollary 11.2. *The set Ω_N is a N -pre-ensemble.*

Proof. It follows from the inequality (11.15) that the property $\|\gamma\| \asymp N$ holds for each matrix $\gamma \in \Omega_N$. By the Corollary 11.1 we have $\#\Omega_N \geq_\epsilon N^{2\delta - \epsilon}$. To put it another way, both items of the definition of pre-ensemble hold. The corollary is proved. \square

We now verify the last property of ensemble related to the relations (7.10) and (7.11).

Lemma 11.5. *For any M in the interval (9.12) there exist j and h in the intervals (9.13), such that for any collection of matrices (11.2) the following inequalities hold*

$$0,99\|\xi_1\xi_2\ldots\xi_j\|^{1-\epsilon_0} \leq M \leq 70A^2\|\xi_1\xi_2\ldots\xi_j\|, \quad (11.39)$$

$$\frac{1}{73A^2}\|\xi_{h+1}\xi_{h+2}\ldots\xi_{2J+1}\|^{1-\epsilon_0} \leq M \leq 150A^2\|\xi_{h+1}\xi_{h+2}\ldots\xi_{2J+1}\|. \quad (11.40)$$

Proof. Let M be fixed in the interval (9.12). Then using Lemma 9.3 we find j and h in (9.13), such that the inequalities (9.14) and (9.15) hold. We note that a bilateral bound on the number N_{j-J} in terms of $\|\xi_1\xi_2\ldots\xi_j\|$ follows from (11.14). To obtain the inequality (11.39) one should substitute this bilateral bound into (9.14). To obtain the inequality (11.40) one should substitute a bilateral bound on N/N_{h-J} , following from the inequality (11.17) with $j = h$, into (9.15). This completes the proof of the lemma. \square

Corollary 11.3. *For any M , satisfying the inequality*

$$\exp\left(\frac{10^5 A^4}{\epsilon_0^2}\right) \leq M \leq N \exp\left(-\frac{10^5 A^4}{\epsilon_0^2}\right), \quad (11.41)$$

there exist indexes j and h in the intervals (9.13), such that for any collection of matrices (11.2) the inequalities (11.39) and (11.40) hold.

Proof. By applying Lemma 9.2, it follows from the inequality (11.41) that the inequality (9.12) holds. It is sufficient now to apply Lemma 11.5. The corollary is proved. \square

Theorem 11.2. *The set Ω_N is a bilateral (ϵ_0, N) -ensemble.*

Proof. The unique expansion of the products, which are equal to Ω_N , has been proved in the beginning of §11. The property of Ω_N to be a N -pre-ensemble has been proved in the Corollary 11.2. The right (ϵ_0, N) -ensemble property is proved by the inequality (11.40), the left -by (11.39), since it follows from the Corollary 11.3 that these inequalities hold for any M , satisfying the inequality (11.41). This completes the proof of the theorem. \square

Thus the main purpose of the section is achieved. However, we formulate a few more properties of a bilateral ensemble. These properties will be of use while estimating exponential sums.

For M , satisfying the inequality (11.41), we denote by \hat{j} and \hat{h} the numbers j and h from Lemma 9.3 corresponding to M . For brevity we will write that numbers \hat{j} , \hat{h} corresponds to M . In the following theorem $\hat{j}^{(1)}$ corresponds to $M^{(1)}$, and $\hat{h}^{(3)}$ corresponds to $M^{(3)}$.

Lemma 11.6. *Let the inequality*

$$M^{(1)}M^{(3)} < N^{1-\epsilon_0}. \quad (11.42)$$

holds for $M^{(1)}$ and $M^{(3)}$ in the interval (11.41). Then $\hat{j}^{(1)} < \hat{h}^{(3)}$, and for $M = M^{(1)}$ the inequality (11.39) holds and for $M = M^{(3)}$ the inequality (11.40) holds.

Proof. It is sufficient to verify the condition $\hat{j}^{(1)} < \hat{h}^{(3)}$. Then the statement of Lemma 11.6 will follow from Lemma 11.5 and Corollary 11.3.

We recall that $\hat{h}^{(3)} = 2J - \hat{j}^{(3)} + 1$. Hence, it is sufficient to prove that

$$\hat{j}^{(1)} + \hat{j}^{(3)} < 2J + 1.$$

Assume the contrary. It follows from (9.16) that

$$N_{\hat{j}^{(1)}-1-J}N_{\hat{j}^{(3)}-1-J} \leq M^{(1)}M^{(3)}.$$

Since $\hat{j}^{(1)} + \hat{j}^{(3)} \geq 2J + 1$, one can consider the inequality $\hat{j}^{(1)} \geq J + 1$ to hold. It follows from the relation $\hat{j}^{(3)} - 1 - J \geq -\hat{j}^{(1)} + J$ and the increasing of the sequence $\{N_j\}_{j=-J-1}^{J+1}$, that

$$N_{\hat{j}^{(1)}-1-J}N_{-\hat{j}^{(1)}+J} \leq M^{(1)}M^{(3)}, \quad \hat{j}^{(1)} \geq J + 1.$$

We put $m = \hat{j}^{(1)} - J - 1$, then

$$N_m N_{-m-1} \leq M^{(1)}M^{(3)}, \quad m \geq 0.$$

Because of (9.4), for $m \geq 0$ we obtain

$$N_m N_{-m-1} = N^{1-\frac{1}{2-\epsilon_0}(1-\epsilon_0)^m + \frac{1}{2-\epsilon_0}(1-\epsilon_0)^{m+2}} = N^{1-\epsilon_0(1-\epsilon_0)^m} \geq N^{1-\epsilon_0}.$$

Hence, the inequality $M^{(1)}M^{(3)} \geq N^{1-\epsilon_0}$ holds, contrary to (11.42). This completes the proof of the lemma. \square

We write

$$j_0(M) = j_0(0, \hat{j}) = j_0(\hat{h}, 2J+1), \quad \Omega_2(M) = \Omega(\hat{h}, 2J+1), \quad \Omega_1(M) = \Omega(0, \hat{j}), \quad (11.43)$$

where $j_0(j, h)$ is defined in (9.19) and $\Omega(j, h)$ is defined in (11.21). Let verify that $j_0(M)$ is well-defined. Actually, since $\hat{h} = 2J - \hat{j} + 1$, so we are to prove that

$$j_0(0, \hat{j}) = j_0(2J - \hat{j} + 1, 2J+1) \text{ for } 2 \leq \hat{j} \leq 2J.$$

If $\hat{j} \leq J+1$, then

$$j_0(0, \hat{j}) = \left| \hat{j} - J - 1 \right| = J + 1 - \hat{j} = \left| 2J + 2 - \hat{j} - J - 1 \right| = j_0(2J - \hat{j} + 1, 2J+1).$$

If $\hat{j} > J+1$, then

$$j_0(0, \hat{j}) = j_0(2J - \hat{j} + 1, 2J+1) = 0.$$

Hence, $j_0(M)$ is well-defined.

Theorem 11.3. *For any M , satisfying the inequality (11.41), the ensemble Ω_N can be represented in two ways может быть представлен двояким образом в виде*

$$\Omega_N = \Omega^{(1)}\Omega^{(2)} \quad \text{or} \quad \Omega_N = \Omega^{(4)}\Omega^{(3)}, \quad (11.44)$$

where

$$\begin{aligned} \Omega^{(1)} &= \Omega_1(M) = \Xi_1 \Xi_2 \dots \Xi_{\hat{j}}, & \Omega^{(2)} &= \Xi_{\hat{j}+1} \Xi_{j+2} \dots \Xi_{2J+1} \\ \Omega^{(3)} &= \Omega_2(M) = \Xi_{\hat{h}+1} \Xi_{\hat{h}+2} \dots \Xi_{2J+1}, & \Omega^{(4)} &= \Xi_1 \Xi_2 \dots \Xi_{\hat{h}} \end{aligned}$$

and for any $\gamma_1 \in \Omega^{(1)}$, $\gamma_2 \in \Omega^{(2)}$, $\gamma_3 \in \Omega^{(3)}$ the following inequalities hold

$$\frac{M}{70A^2} \leq \|\gamma_1\| \leq 1,03M^{1+2\epsilon_0}, \quad (11.45)$$

$$\frac{N}{140A^2 H_1(M)} \leq \|\gamma_2\| \leq \frac{73A^2 N}{M}, \quad \text{where} \quad H_1(M) = 1,03M^{1+2\epsilon_0}, \quad (11.46)$$

$$\frac{M}{150A^2} \leq \|\gamma_3\| \leq H_3(M), \quad \text{where} \quad H_3(M) = 80A^{2,1}M^{1+2\epsilon_0}. \quad (11.47)$$

Proof. In view of the Corollary 11.3 the inequalities (11.39) and (11.40) hold for the matrices $\gamma_1 = \xi_1 \xi_2 \dots \xi_{\hat{j}}$ and $\gamma_3 = \xi_{\hat{h}+1} \xi_2 \dots \xi_{2J+1}$. Using these inequalities we obtain (11.45) and (11.47). Moreover, it follows from (5.4) and (11.15) that

$$\frac{N}{140A^2} \leq \frac{\|\xi_1 \xi_2 \dots \xi_{2J+1}\|}{2} \leq \|\gamma_1\| \|\gamma_2\| \leq \|\xi_1 \xi_2 \dots \xi_{2J+1}\| \leq 1,01N. \quad (11.48)$$

Substituting the bound (11.45) into (11.48) we obtain the inequality (11.46). This completes the proof of the theorem. \square

Theorem 11.4. Let $M^{(2)} \geq (M^{(1)})^{2\epsilon_0}$, and let the inequality (11.41) holds for $M = M^{(1)}$ and $M = M^{(1)}M^{(2)}$. Then the ensemble Ω_N can be represented in the following form

$$\Omega_N = \Omega^{(1)}\Omega^{(2)}\Omega^{(3)},$$

where

$$\begin{aligned}\Omega^{(1)} &= \Omega_1(M^{(1)}) = \Xi_1\Xi_2\cdots\Xi_{\hat{j}_1}, & \Omega^{(2)} &= \Xi_{\hat{j}_1+1}\Xi_{\hat{j}_1+2}\cdots\Xi_{\hat{j}_2} \\ \Omega^{(1)}\Omega^{(2)} &= \Omega_1(M^{(1)}M^{(2)}), \\ \Omega^{(3)} &= \Xi_{\hat{j}_2+1}\Xi_{\hat{j}_2+2}\cdots\Xi_{2J+1},\end{aligned}$$

where \hat{j}_1 corresponds to $M^{(1)}$, and \hat{j}_2 corresponds to $M^{(1)}M^{(2)}$. And for any $\gamma_1 \in \Omega^{(1)}$, $\gamma_2 \in \Omega^{(2)}$, $\gamma_3 \in \Omega^{(3)}$ the following inequalities hold

$$\frac{M^{(1)}}{70A^2} \leq \|\gamma_1\| \leq 1, 03(M^{(1)})^{1+2\epsilon_0}, \quad (11.49)$$

$$\frac{M^{(1)}M^{(2)}}{70A^2} \leq \|\gamma_1\gamma_2\| \leq 1, 03(M^{(1)}M^{(2)})^{1+2\epsilon_0}, \quad (11.50)$$

$$\frac{M^{(2)}}{150A^2(M^{(1)})^{2\epsilon_0}} \leq \|\gamma_2\| \leq 73A^2 \frac{(M^{(2)})^{1+2\epsilon_0}}{(M^{(1)})^{2\epsilon_0}}, \quad (11.51)$$

$$\frac{N}{150A^2H_1(M^{(1)}M^{(2)})} \leq \|\gamma_3\| \leq \frac{73A^2N}{M^{(1)}M^{(2)}}. \quad (11.52)$$

Proof. To ensure that the partition of the ensemble is well-defined, it is enough to verify that $\hat{j}_2 > \hat{j}_1$. To do this it is sufficient to prove that for any $M^{(1)}$ in the interval $N_{\hat{j}_1-1-J} \leq M^{(1)} \leq N_{\hat{j}_1-J}$ the inequality $N_{\hat{j}_1-J} \leq M^{(1)}M^{(2)}$ holds. It follows from the conditions of the theorem that

$$M^{(2)} \geq (M^{(1)})^{2\epsilon_0} \geq (M^{(1)})^{\frac{1}{1-\epsilon_0}-1} \geq \frac{N_{\hat{j}_1-J}}{M^{(1)}}, \quad (11.53)$$

where the last inequality holds because of $M^{(1)} \geq N_{\hat{j}_1-J}^{1-\epsilon_0}$ see (9.14). Thus we have proved that the partition of the ensemble is well-defined. The bounds (11.49) and (11.50) follow from (11.45) and (11.52) follows from (11.46). Finally, (11.51) follows from (11.49), (11.50) and (5.4). This completes the proof of the theorem. \square

Theorem 11.5. For any $M^{(1)}$ and $M^{(3)}$ satisfying the hypotheses of Lemma 11.6 the ensemble Ω_N can be represented in the following form

$$\Omega_N = \Omega^{(1)}\Omega^{(2)}\Omega^{(3)},$$

where

$$\begin{aligned}\Omega^{(1)} &= \Omega_1(M^{(1)}) = \Xi_1\Xi_2\cdots\Xi_{\hat{j}_1}, & \Omega^{(2)} &= \Xi_{\hat{j}_1+1}\Xi_{\hat{j}_1+2}\cdots\Xi_{\hat{h}_2}, \\ \Omega^{(3)} &= \Omega_2(M^{(2)}) = \Xi_{\hat{h}_2+1}\Xi_{\hat{h}_2+2}\cdots\Xi_{2J+1},\end{aligned}$$

and for any $\gamma_1 \in \Omega^{(1)}$, $\gamma_2 \in \Omega^{(2)}$, $\gamma_3 \in \Omega^{(3)}$ the following inequalities hold

$$\frac{M^{(1)}}{70A^2} \leq \|\gamma_1\| \leq 1,03(M^{(1)})^{1+2\epsilon_0}, \quad (11.54)$$

$$\frac{N}{3 \cdot 10^4 A^5 (M^{(1)} M^{(3)})^{1+2\epsilon_0}} \leq \|\gamma_2\| \leq \frac{11000 A^4 N}{M^{(1)} M^{(3)}}, \quad (11.55)$$

$$\frac{M^{(3)}}{150A^2} \leq \|\gamma_3\| \leq 80A^{2,1}(M^{(3)})^{1+2\epsilon_0}. \quad (11.56)$$

Proof. The existence of the partition is ensured by the inequality $\hat{j}_1 < \hat{h}_3$ in Lemma 11.6 ($\hat{j}_1 = j^{(1)}$, $\hat{h}_3 = h^{(3)}$). It also follows from Lemma 11.6 that the inequalities (11.39) and (11.40) hold for $\gamma_1 \in \Omega^{(1)}$ and $\gamma_3 \in \Omega^{(3)}$. Using these inequalities we obtain the estimates (11.54) and (11.56). Next, in the same way as (11.48) we obtain

$$\frac{N}{280A^2} \leq \|\gamma_1\| \|\gamma_2\| \|\gamma_3\| \leq 1,01N. \quad (11.57)$$

Substituting (11.54) and (11.56) into (11.57), we obtain (11.55). This completes the proof of the theorem. \square

Lemma 11.7. For M in the interval (11.41) the following inequality holds

$$\frac{\log \log M^2}{\log(1 - \epsilon_0)} - 1 \leq j_0(M) + \frac{\log \log N}{\log(1 - \epsilon_0)}. \quad (11.58)$$

Proof. We consider two cases.

1. Let $\exp\left(\frac{10^5 A^4}{\epsilon_0^2}\right) \leq M \leq N_1 = N^{\frac{1}{2-\epsilon_0}}$.

Then it follows from Lemma 9.3 and (9.16) that firstly $2 \leq \hat{j} \leq J+1$ and secondly

$$N_{\hat{j}-J}^{1-\epsilon_0} \leq M \leq N_{\hat{j}-J}.$$

Hence $j_0(M) = J+1 - \hat{j}$ and applying (9.4) we obtain

$$M \geq N_{\hat{j}-J}^{1-\epsilon_0} = N^{\frac{1}{2-\epsilon_0}(1-\epsilon_0)^{2-\hat{j}+J}} = N^{\frac{1}{2-\epsilon_0}(1-\epsilon_0)^{1+j_0(M)}} \geq N^{\frac{1}{2}(1-\epsilon_0)^{1+j_0(M)}}. \quad (11.59)$$

Taking a logarithm twice we obtain

$$\log \log M^2 \geq (1 + j_0(M)) \log(1 - \epsilon_0) + \log \log N.$$

From this the inequality (11.58) follows.

2. Let $N^{\frac{1}{2-\epsilon_0}} = N_1 \leq M \leq N \exp\left(-\frac{10^5 A^4}{\epsilon_0^2}\right)$.

Then $\hat{j} > J+1$ and, hence, $j_0(M) = 0$. In view of Lemma 9.2 and (9.4) we obtain

$$\begin{aligned} M &\geq N_{\hat{j}-J}^{1-\epsilon_0} = N^{\left(1-\frac{1}{2-\epsilon_0}(1-\epsilon_0)^{\hat{j}-J}\right)(1-\epsilon_0)} \geq N^{\left(1-\frac{1}{2-\epsilon_0}(1-\epsilon_0)^2\right)(1-\epsilon_0)} \\ &\geq N^{\frac{1}{2}(1-\epsilon_0)^{1+j_0(M)}}, \end{aligned} \quad (11.60)$$

since $1 - \frac{1}{2-\epsilon_0}(1-\epsilon_0)^2 \geq \frac{1}{2}$. The inequality (11.60) coincides with the bound (11.59). Thus we obtain (11.58) in the same way.

The lemma is proved. □

For M in the interval (11.41) we define the following function

$$T(M) = \exp \left(- \left(\frac{\log \log M^2}{\log(1 - \epsilon_0)} - 1 \right)^2 \right). \quad (11.61)$$

We observe that for $\epsilon_0 \in (0, \frac{1}{2500})$ one has

$$M^{-\epsilon_0} \leq T(M), \quad \text{if } M \geq \exp(\epsilon_0^{-5}). \quad (11.62)$$

Theorem 11.6. *Let $M \geq \exp(\epsilon_0^{-5})$ and belongs to the interval (11.41). Then the following bounds on the cardinality of the sets $|\Omega_1(M)|$ and $|\Omega_2(M)|$ hold*

$$M^{2\delta - \epsilon_0} \leq M^{2\delta} T(M) \leq |\Omega_1(M)| \leq 9M^{2\delta + 4\epsilon_0}, \quad (11.63)$$

$$|\Omega_2(M)| \geq M^{2\delta} T(M) \geq M^{2\delta - \epsilon_0}. \quad (11.64)$$

Proof. Using the definition of the set $|\Omega_1(M)|$ and the inequality

$$N_{\hat{j}-J} \leq M^{\frac{1}{1-\epsilon_0}} \leq M^{1+2\epsilon_0}$$

we obtain that the upper bound in (11.63) follows immediately from Lemma 11.3.

1. We estimate the cardinality of the set $|\Omega_1(M)|$ from below. Taking into account that $M \leq N_{\hat{j}-J}$ and $j_0(M) = j_0(0, \hat{j})$, it follows from the Theorem 11.1 that

$$|\Omega_1(M)| \geq M^{2\delta} \exp \left(- \left(\frac{\log \log N}{\log(1 - \epsilon_0)} + j_0(M) \right)^2 \right). \quad (11.65)$$

Using (11.34) and (11.58), we obtain, in view of $j_0(M) \leq J$, that

$$\frac{\log \log M^2}{\log(1 - \epsilon_0)} - 1 \leq j_0(M) + \frac{\log \log N}{\log(1 - \epsilon_0)} \leq 0.$$

Hence,

$$T(M) \leq \exp \left(- \left(\frac{\log \log N}{\log(1 - \epsilon_0)} + j_0(M) \right)^2 \right).$$

Substituting the estimate into (11.65), we obtain the lower bound in (11.63).

2. We estimate the cardinality of the set $|\Omega_2(M)|$ from below. Taking into account that $M \leq \frac{N}{N_{\hat{h}-J}}$ and $j_0(M) = j_0(\hat{h}, 2J + 1)$, it follows from the Theorem 11.1 that

$$|\Omega_2(M)| \geq M^{2\delta} \exp \left(- \left(\frac{\log \log N}{\log(1 - \epsilon_0)} + j_0(M) \right)^2 \right). \quad (11.66)$$

From this the estimate (11.64) follows in the same way.

The theorem is proved. □

Chapter III

Estimates of exponential sums and integrals. A generalization of the Bourgain-Kontorovich's method.

12 General estimates of exponential sums over ensemble.

Recall that to prove our main theorem 2.1 we should obtain the maximum accurate bound on the integral

$$\int_0^1 |S_N(\theta)|^2 d\theta = \int_0^1 \left| \sum_{\gamma \in \Omega_N} e(\theta \|\gamma\|) \right|^2 d\theta, \quad (12.1)$$

where N is a sufficiently large integer and for $\|\gamma\|$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & d_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & d_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & d_k \end{pmatrix} \quad (12.2)$$

the norm $\|\gamma\|$ is defined by

$$\|\gamma\| = d = \langle d_1, d_2, \dots, d_k \rangle = (0, 1) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (12.3)$$

Then

$$S_N(\theta) = \sum_{\gamma \in \Omega_N} e(\theta \|\gamma\|) = \sum_{\gamma \in \Omega_N} e((0, 1)\gamma \begin{pmatrix} 0 \\ 1 \end{pmatrix} \theta). \quad (12.4)$$

To estimate the sum (12.4) different methods were used in [1] depending on the value of θ . The basis of all these (and even new one) methods can be presented in an unified manner if special notations are used. Suppose that a partition of the ensemble

$$\Omega_N = \Omega^{(1)}\Omega^{(2)}\Omega^{(3)} \quad (12.5)$$

is given and that this partition has one of the following forms:

- or as in Theorem 11.5 and then

$$\Omega^{(1)} = \Omega_1(M^{(1)}), \quad \Omega^{(3)} = \Omega_2(M^{(3)}), \quad (12.6)$$

- or as in Theorem 11.4 and then

$$\Omega^{(1)}\Omega^{(2)} = \Omega_1(M^{(1)}M^{(2)}), \quad (12.7)$$

- or as in Theorem 11.3 $\Omega_N = \Omega^{(1)}\Omega^{(3)}$, and then

$$\Omega^{(1)} = \Omega_1(M^{(1)}), \quad \Omega^{(2)} = \{E\}, \quad (12.8)$$

where E – is a unit matrix 2×2 , and $M^{(1)} = M$.

We recall that

$$H_1(M) = 1,03M^{1+2\epsilon_0}, \quad H_3(M) = 80A^{2,1}M^{1+2\epsilon_0} \quad (12.9)$$

and for $\gamma_1 \in \Omega^{(1)}$ one has

$$\|\gamma_1\| \leq H_1(M^{(1)}) = H_1. \quad (12.10)$$

For $n \in \{1, 2, 3\}$ we write

$$\tilde{\Omega}^{(n)} = \begin{cases} (0, 1)\Omega^{(1)} = \left\{ (0, 1)g_1 \mid g_1 \in \Omega^{(1)} \right\}, & \text{если } n = 1, \\ \Omega^{(2)}, & \text{если } n = 2, \\ \Omega^{(3)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \left\{ g_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mid g_3 \in \Omega^{(3)} \right\}, & \text{если } n = 3. \end{cases}, \quad (12.11)$$

Let define the following function

$$T(x) = \max \{0, 1 - |x|\}, \quad S(x) = \left(\frac{\sin \pi x}{\pi x} \right)^2. \quad (12.12)$$

It is common knowledge [6, (4.83)] that $\hat{S}(x) = T(x)$, where $\hat{f}(x) = \int_{-\infty}^{\infty} f(t)e(-xt)dt$ is the Fourier transform of the function $f(x)$. We consider $S_2(x) = 3S(\frac{x}{2})$. It is obvious that $S_2(x)$ is a nonnegative function and

$$S_2(x) > 1 \text{ for } x \in [-1, 1]. \quad (12.13)$$

Since $\hat{S}_2(x) = 6T(2x)$, we have $\hat{S}_2(x) \neq 0$ only for $|x| < \frac{1}{2}$. We next undertake to estimate the sum of the form

$$\sigma_{N,Z} = \sum_{\theta \in Z} |S_N(\theta)|, \quad (12.14)$$

where Z is a finite subset of the interval $[0, 1]$.

Lemma 12.1. *For either $(\mu, \lambda) = (2, 3)$ or $(\mu, \lambda) = (3, 2)$ the following estimate holds*

$$\sigma_{N,Z} \leq |\Omega^{(1)}|^{1/2} \sum_{g_\lambda \in \tilde{\Omega}^{(\lambda)}} \left(\sum_{g_1 \in \mathbb{Z}^2} \mathbb{S} \left(\frac{g_1}{H_1} \right) \left| \sum_{\theta \in Z} \xi(\theta) \sum_{g_\mu \in \tilde{\Omega}^{(\mu)}} e(g_1 g_2 g_3 \theta) \right| \right)^{1/2}, \quad (12.15)$$

where $\mathbb{S}(x, y) = S_2(x)S_2(y)$, and $\xi(\theta)$ is a complex number with $|\xi(\theta)| = 1$.

Proof. The numbers $\xi(\theta)$ are defined by the relation $|S_N(\theta)| = \xi(\theta)S_N(\theta)$. Then we obtain from the formulae (12.4), (12.5) and the definition (12.11) that

$$\sigma_{N,Z} = \sum_{\theta \in Z} \xi(\theta)S_N(\theta) = \sum_{\theta \in Z} \xi(\theta) \sum_{g_1 \in \tilde{\Omega}^{(1)}} \sum_{g_2 \in \tilde{\Omega}^{(2)}} \sum_{g_3 \in \tilde{\Omega}^{(3)}} e(g_1 g_2 g_3 \theta), \quad (12.16)$$

where g_1, g_3 are already vectors in \mathbb{Z}^2 . It follows from (12.16) that

$$\sigma_{N,Z} \leq \sum_{g_\lambda \in \tilde{\Omega}^{(\lambda)}} \sum_{\substack{g_1 \in \mathbb{Z}^2 \\ |g_1| \leq H_1}} \mathbf{1}_{g_1 \in \tilde{\Omega}^{(1)}} \left| \sum_{\theta \in Z} \xi(\theta) \sum_{g_\mu \in \tilde{\Omega}^{(\mu)}} e(g_1 g_2 g_3 \theta) \right|. \quad (12.17)$$

We note that in view of (12.10) the condition $|g_1| \leq H_1$ does not impose any extra restrictions. Applying the Cauchy-Schwarz inequality in (12.17) we obtain

$$\sigma_{N,Z} \leq \sum_{g_\lambda \in \tilde{\Omega}^{(\lambda)}} \left(\sum_{\substack{g_1 \in \mathbb{Z}^2 \\ |g_1| \leq H_1}} \mathbf{1}_{g_1 \in \tilde{\Omega}^{(1)}} \right)^{1/2} \left(\sum_{\substack{g_1 \in \mathbb{Z}^2 \\ |g_1| \leq H_1}} \left| \sum_{\theta \in Z} \xi(\theta) \sum_{g_\mu \in \tilde{\Omega}^{(\mu)}} e(g_1 g_2 g_3 \theta) \right|^2 \right)^{1/2}. \quad (12.18)$$

Considering that

$$\left(\sum_{\substack{g_1 \in \mathbb{Z}^2 \\ |g_1| \leq H_1}} \mathbf{1}_{g_1 \in \tilde{\Omega}^{(1)}} \right)^{1/2} = |\Omega^{(1)}|^{1/2}$$

and taking into account that the function $\mathbb{S}(x, y) > 1$ for $(x, y) \in [-1, 1]^2$ and is nonnegative, we obtain (12.15). This completes the proof of the lemma. \square

To reduce our notations we in what follows will identify any variable x with $x^{(1)}$.

Lemma 12.2. *Under the hypotheses of Lemma 12.1 the following bound holds*

$$\sigma_{N,Z} \leq 10H_1 |\Omega^{(1)}|^{1/2} \sum_{g_\lambda \in \tilde{\Omega}^{(\lambda)}} \left(\sum_{\substack{g_\mu^{(1)}, g_\mu^{(2)} \in \tilde{\Omega}^{(\mu)} \\ \theta^{(1)}, \theta^{(2)} \in Z}} \mathbf{1}_{\{\|z\|_{1,2} \leq \frac{1}{2H_1}\}} \right)^{1/2}, \quad (12.19)$$

where $\|x\|_{1,2} = \max\{\|x_1\|, \|x_2\|\}$ for $x = (x_1, x_2) \in \mathbb{R}^2$, and

$$z = \begin{cases} g_2^{(1)} g_3 \theta^{(1)} - g_2^{(2)} g_3 \theta^{(2)} & \text{if } \mu = 2, \\ g_2 g_3^{(1)} \theta^{(1)} - g_2 g_3^{(2)} \theta^{(2)}, & \text{if } \mu = 3. \end{cases} \quad (12.20)$$

Proof. We note that the quantity z can be represented in a shorter form

$$z = g_2^{(1)} g_3^{(1)} \theta^{(1)} - g_2^{(4-\mu)} g_3^{(\mu-1)} \theta^{(2)} \quad (12.21)$$

Applying the relation $|x|^2 = x\bar{x}$ and reversing orders we easily obtain that

$$\sum_{g_1 \in \mathbb{Z}^2} \mathbb{S}\left(\frac{g_1}{H_1}\right) \left| \sum_{\theta \in Z} \xi(\theta) \sum_{g_\mu \in \tilde{\Omega}^{(\mu)}} e(g_1 g_2 g_3 \theta) \right|^2 \leq \sum_{g_\mu^{(1)}, g_\mu^{(2)} \in \tilde{\Omega}^{(\mu)}} \sum_{\theta^{(1)}, \theta^{(2)} \in Z} \left| \sum_{g_1 \in \mathbb{Z}^2} \mathbb{S}\left(\frac{g_1}{H_1}\right) e(g_1 z) \right|. \quad (12.22)$$

By application of the Poisson summation formula [6, §4.3.]:

$$\sum_{n \in \mathbb{Z}^2} f(n) = \sum_{k \in \mathbb{Z}^2} \hat{f}(k),$$

and writing $f(n) = \mathbb{S}\left(\frac{n}{H_1}\right) e(nz)$, we transform the inner sum in the right side of (12.22):

$$\sum_{g_1 \in \mathbb{Z}^2} \mathbb{S}\left(\frac{g_1}{H_1}\right) e(g_1 z) = \sum_{k \in \mathbb{Z}^2} \int_{x \in \mathbb{R}^2} \mathbb{S}\left(\frac{x}{H_1}\right) e(x(z-k)) dx = H_1^2 \sum_{k \in \mathbb{Z}^2} \hat{\mathbb{S}}((k-z)H_1). \quad (12.23)$$

We note that the relation (12.23) can be obtained directly from [6, (4.25)]. As $\hat{\mathbb{S}}(x, y) \neq 0$ only if $|x| \leq \frac{1}{2}$ and $|y| \leq \frac{1}{2}$, so the sum in the right side of (12.23) consists of at most one summand, hence,

$$\left| \sum_{g_1 \in \mathbb{Z}^2} \mathbb{S} \left(\frac{g_1}{H_1} \right) e(g_1 z) \right| \leq 36 H_1^2 \mathbf{1}_{\{\|z\|_{1,2} \leq \frac{1}{2H_1}\}}. \quad (12.24)$$

Substituting (12.24) into (12.22), we obtain

$$\sum_{g_1 \in \mathbb{Z}^2} \mathbb{S} \left(\frac{g_1}{H_1} \right) \left| \sum_{\theta \in Z} \xi(\theta) \sum_{g_\mu \in \tilde{\Omega}^{(\mu)}} e(g_1 g_2 g_3 \theta) \right|^2 \leq 36 H_1^2 \sum_{\substack{g_\mu^{(1)}, g_\mu^{(2)} \in \tilde{\Omega}^{(\mu)} \\ \theta^{(1)}, \theta^{(2)} \in Z}} \mathbf{1}_{\{\|z\|_{1,2} \leq \frac{1}{2H_1}\}} \quad (12.25)$$

Substituting (12.25) into (12.15), we obtain (12.19). This completes the proof of the lemma. \square

To transform the right side of (12.19), we are to specify the set Z . It follows from the Dirichlet theorem that for any $\theta \in [0, 1]$ there exist $a, q \in \mathbb{N} \cup \{0\}$ and $\beta \in \mathbb{R}$, such that

$$\theta = \frac{a}{q} + \beta, \quad (a, q) = 1, \quad 0 \leq a \leq q \leq N^{1/2}, \quad \beta = \frac{K}{N}, \quad |K| \leq \frac{N^{1/2}}{q}, \quad (12.26)$$

and $a = 0$ or $a = q$ only if $q = 1$. We denote

$$P_{Q_1, Q}^{(\beta)} = \left\{ \theta = \frac{a}{q} + \beta \mid (a, q) = 1, \quad 0 \leq a \leq q, \quad Q_1 \leq q \leq Q \right\}. \quad (12.27)$$

In what follows we always have $Z \subseteq P_{Q_1, Q}^{(\beta)}$ for some Q_1, Q, β . We will write numbers $\theta^{(1)}, \theta^{(2)} \in P_{Q_1, Q}^{(\beta)}$ in the following way

$$\theta^{(1)} = \frac{a^{(1)}}{q^{(1)}} + \beta, \quad \theta^{(2)} = \frac{a^{(2)}}{q^{(2)}} + \beta. \quad (12.28)$$

Let

$$\mathfrak{N}_0(g_\lambda) = \left\{ (g_\mu^{(1)}, g_\mu^{(2)}, \theta^{(1)}, \theta^{(2)}) \in \tilde{\Omega}^{(\mu)} \times \tilde{\Omega}^{(\mu)} \times Z^2 \mid \left\| g_2 g_3 \theta^{(1)} - g_2^{(4-\mu)} g_3^{(\mu-1)} \theta^{(2)} \right\|_{1,2} \leq \frac{1}{2H_1} \right\} \quad (12.29)$$

$$\mathfrak{N}(g_\lambda) = \left\{ (g_\mu^{(1)}, g_\mu^{(2)}, \theta^{(1)}, \theta^{(2)}) \in \tilde{\Omega}^{(\mu)} \times \tilde{\Omega}^{(\mu)} \times Z^2 \mid (12.31) \text{ и } (12.32) \text{ hold} \right\}, \quad (12.30)$$

where

$$\left\| g_2 g_3 \frac{a^{(1)}}{q^{(1)}} - g_2^{(4-\mu)} g_3^{(\mu-1)} \frac{a^{(2)}}{q^{(2)}} \right\|_{1,2} \leq \frac{74 A^2 \overline{K}}{M^{(1)}}, \quad (12.31)$$

$$\left| g_2 g_3 - g_2^{(4-\mu)} g_3^{(\mu-1)} \right|_{1,2} \leq \min \left\{ \frac{73 A^2 N}{M^{(1)}}, \frac{73 A^2 N}{M^{(1)} \overline{K}} + \frac{N}{\overline{K}} \left\| g_2 g_3 \frac{a^{(1)}}{q^{(1)}} - g_2^{(4-\mu)} g_3^{(\mu-1)} \frac{a^{(2)}}{q^{(2)}} \right\|_{1,2} \right\}, \quad (12.32)$$

and $\overline{K} = \max\{1, |K|\}$. We note that it follows from Lemma 12.2 and the definition (12.29) that

$$\sigma_{N,Z} \leq 10H_1 |\Omega^{(1)}|^{1/2} \sum_{g_\lambda \in \tilde{\Omega}^{(\lambda)}} |\mathfrak{N}_0(g_\lambda)|^{1/2}. \quad (12.33)$$

Lemma 12.3. *For $Z \subseteq P_{Q_1, Q}^{(\beta)}$ and for $M^{(1)}$ in (11.41) such that $M^{(1)} > 146A^2\overline{K}$, the following inequality holds*

$$\sigma_{N,Z} \leq 10H_1 |\Omega^{(1)}|^{1/2} \sum_{g_\lambda \in \tilde{\Omega}^{(\lambda)}} |\mathfrak{N}(g_\lambda)|^{1/2}. \quad (12.34)$$

Proof. In view of (12.33), it is sufficient to prove that $\mathfrak{N}_0(g_\lambda) \subseteq \mathfrak{N}(g_\lambda)$. Let $\Omega = \Omega^{(2)}\Omega^{(3)}$, then $\Omega_N = \Omega^{(1)}\Omega$. For any partition (12.6), (12.7), (12.8) of the ensemble Ω_N using (5.4) and the lower bound on $\|\gamma_1\|$ from the theorem corresponding to the partition, we obtain

$$|g_2g_3 - g_2^{(4-\mu)}g_3^{(\mu-1)}|_{1,2} \leq \max_{\gamma \in \Omega} \|\gamma\| \leq \frac{73A^2N}{M^{(1)}}. \quad (12.35)$$

Hence,

$$|(g_2g_3 - g_2^{(4-\mu)}g_3^{(\mu-1)})\beta|_{1,2} \leq \frac{73A^2N\overline{K}}{M^{(1)}N} \leq \frac{1}{2}, \quad (12.36)$$

so

$$|(g_2g_3 - g_2^{(4-\mu)}g_3^{(\mu-1)})\beta|_{1,2} = \|(g_2g_3 - g_2^{(4-\mu)}g_3^{(\mu-1)})\beta\|_{1,2}, \quad (12.37)$$

and

$$\|(g_2g_3 - g_2^{(4-\mu)}g_3^{(\mu-1)})\beta\|_{1,2} \leq \frac{73A^2\overline{K}}{M^{(1)}}. \quad (12.38)$$

It follows from the definition (12.29) and the bound (12.38) that

$$\|g_2g_3 \frac{a^{(1)}}{q^{(1)}} - g_2^{(4-\mu)}g_3^{(\mu-1)} \frac{a^{(2)}}{q^{(2)}}\|_{1,2} \leq \frac{73A^2\overline{K}}{M^{(1)}} + \frac{1}{M^{(1)}} < \frac{74A^2\overline{K}}{M^{(1)}}, \quad (12.39)$$

that is, for $(g_\mu^{(1)}, g_\mu^{(2)}, \theta^{(1)}, \theta^{(2)}) \in \mathfrak{N}_0(g_\lambda)$ the inequality (12.31) holds. Using (12.37) and (12.29), we obtain

$$|(g_2g_3 - g_2^{(4-\mu)}g_3^{(\mu-1)})\frac{K}{N}|_{1,2} \leq \frac{1}{H_1} + \|g_2g_3 \frac{a^{(1)}}{q^{(1)}} - g_2^{(4-\mu)}g_3^{(\mu-1)} \frac{a^{(2)}}{q^{(2)}}\|_{1,2}, \quad (12.40)$$

whence

$$|g_2g_3 - g_2^{(4-\mu)}g_3^{(\mu-1)}|_{1,2} \leq \frac{N}{M^{(1)}|K|} + \frac{N}{|K|} \left\| g_2g_3 \frac{a^{(1)}}{q^{(1)}} - g_2^{(4-\mu)}g_3^{(\mu-1)} \frac{a^{(2)}}{q^{(2)}} \right\|_{1,2}. \quad (12.41)$$

The inequality (12.32) follows from the estimates (12.41) and (12.35). This completes the proof of the lemma. \square

We denote

$$\mathfrak{M}(g_\lambda) = \left\{ (g_\mu^{(1)}, g_\mu^{(2)}, \theta^{(1)}, \theta^{(2)}) \in \tilde{\Omega}^{(\mu)} \times \tilde{\Omega}^{(\mu)} \times Z^2 \mid (12.43) \text{ и } (12.44) \text{ hold} \right\}, \quad (12.42)$$

where

$$|g_2 g_3 - g_2^{(4-\mu)} g_3^{(\mu-1)}|_{1,2} \leq \frac{73A^2 N}{M^{(1)} \overline{K}}, \quad (12.43)$$

$$\left\| g_2 g_3 \frac{a^{(1)}}{q^{(1)}} - g_2^{(4-\mu)} g_3^{(\mu-1)} \frac{a^{(2)}}{q^{(2)}} \right\|_{1,2} = 0. \quad (12.44)$$

Lemma 12.4. *Let the hypotheses of one of the Theorem 11.5, 11.4 or 11.3, on which the partition of Ω_N in the form (12.5) is based, hold. Let $M^{(1)}$ be such that for any $\theta^{(1)}, \theta^{(2)} \in Z$ the following inequality holds*

$$[q^{(1)}, q^{(2)}] < \frac{M^{(1)}}{74A^2 \overline{K}}. \quad (12.45)$$

Then the following bound holds

$$\sigma_{N,Z} \leq 10H_1 \left| \Omega^{(1)} \right|^{1/2} \sum_{g_\lambda \in \tilde{\Omega}^{(\lambda)}} |\mathfrak{M}(g_\lambda)|^{1/2}. \quad (12.46)$$

Proof. In view of (12.34) it is sufficient to prove that $\mathfrak{N}(g_\lambda) \subseteq \mathfrak{M}(g_\lambda)$. We note that to prove this it is sufficient to obtain that under the hypotheses of Lemma 12.4 and $(g_\mu^{(1)}, g_\mu^{(2)}, \theta^{(1)}, \theta^{(2)}) \in \mathfrak{N}_0(g_\lambda)$ the relation (12.44) holds. It follows from (12.31) and (12.45) that

$$\left\| g_2 g_3 \frac{a^{(1)}}{q^{(1)}} - g_2^{(4-\mu)} g_3^{(\mu-1)} \frac{a^{(2)}}{q^{(2)}} \right\|_{1,2} \leq \frac{74A^2 \overline{K}}{M^{(1)}} < \frac{1}{[q^{(1)}, q^{(2)}]}, \quad (12.47)$$

this implies that (12.44). This completes the proof of the lemma. \square

Thus we reduced the problem of estimating $\sigma_{N,Z}$ to the evaluation the cardinality of one of the sets $\mathfrak{N}_0(g_\lambda)$, $\mathfrak{N}(g_\lambda)$, $\mathfrak{M}(g_\lambda)$. The choice of the set will depend on μ . Let state one more lemma of a general nature. A similar statement was used by S.V. Konyagin in [14, 17].

Lemma 12.5. *Let W be a finite subset of the interval $[0, 1]$ and let $|W| > 10$. Let $f : W \rightarrow \mathbb{R}_+$ be a function such that, for any subset $Z \subseteq W$ the following bound holds*

$$\sum_{\theta \in Z} f(\theta) \leq C_1 |Z|^{1/2} + C_2,$$

where C_1, C_2 are non-negative constants not depending on the set Z . Then the following estimate holds

$$\sum_{\theta \in W} f^2(\theta) \ll C_1^2 \log |W| + C_2 \max_{\theta \in W} f(\theta) \quad (12.48)$$

with the absolute constant in Vinogradov symbol.

Proof. Let number the value of $f(\theta)$ in the decreasing order

$$f_1 \geq \dots \geq f_{|W|} > 0.$$

If $C_1 \geq C_2$, in particular, if $C_2 = 0$, we have $\sum_{\theta \in Z} f(n) \leq 2C_1|Z|^{1/2}$. Then for any k such that $1 \leq k \leq |W|$, one has

$$kf_k \leq \sum_{n=1}^k f_n \leq 2C_1 k^{1/2}$$

and, hence, $f_k \leq 2C_1 k^{-1/2}$. Thus

$$\sum_{\theta \in W} f^2(\theta) = \sum_{n=1}^{|W|} f_n^2 \leq 8C_1^2 \log |W|.$$

Therefore, it remains to consider the case $C_1 < C_2$. In a similar manner we obtain that

$$kf_k \leq \sum_{n=1}^k f_n \leq C_1 k^{1/2} + C_2 \Rightarrow f_k \leq \frac{C_1}{k^{1/2}} + \frac{C_2}{k}.$$

Let

$$M = \max_{\theta \in W} f(n), \quad L = \frac{C_2^{4/3}}{(C_1 M)^{2/3}}.$$

We consider three cases.

1. If $L < 10$, then

$$\sum_{\theta \in W} f^2(\theta) = \sum_{k \leq |W|} f_k^2 \ll \sum_{k \leq |W|} \left(\frac{C_1^2}{k} + \frac{C_2^2}{k^2} \right) \ll C_1^2 \log |W| + C_2^2. \quad (12.49)$$

It follows from the condition $L < 10$ that $C_2^2 \ll C_1 M$. Substituting this bound into (12.49) and taking into account $C_1 < C_2$, we obtain

$$\sum_{\theta \in W} f^2(\theta) \ll C_1^2 \log |W| + C_2 M. \quad (12.50)$$

2. If $10 \leq L < |W|$, then

$$\begin{aligned} \sum_{\theta \in W} f^2(\theta) &= \sum_{k \leq L} f_k^2 + \sum_{L < k \leq |W|} f_k^2 \ll \max_{\theta \in W} f(n) \sum_{k \leq L} f_k + \sum_{L < k \leq |W|} \left(\frac{C_1^2}{k} + \frac{C_2^2}{k^2} \right) \ll \\ &\ll (C_1 L^{1/2} + C_2) \max_{\theta \in W} f(\theta) + C_1^2 \log \frac{|W|}{L} + \frac{C_2^2}{L}. \end{aligned} \quad (12.51)$$

It follows from the definition of L that $C_1 L^{1/2} M = \frac{C_2^2}{L}$. Then by (12.51) we obtain

$$\sum_{\theta \in W} f^2(\theta) \ll C_1^2 \log |W| + C_2 M + (C_1 C_2 M)^{2/3} < C_1^2 \log |W| + C_2 M. \quad (12.52)$$

3. If $L \geq |W|$, then

$$\sum_{\theta \in W} f^2(\theta) \ll \max_{\theta \in W} f(n) \sum_{k \leq |W|} f_k \leq (C_1 |W|^{1/2} + C_2)M. \quad (12.53)$$

It follows from $L \geq |W|$ that $|W|^{1/2} \leq \frac{C_2^{2/3}}{(C_1 M)^{1/3}}$. Substituting this bound into (12.54), we obtain

$$\sum_{\theta \in W} f^2(\theta) \ll (C_1 C_2 M)^{2/3} + C_2 M < C_1^2 \log |W| + C_2 M. \quad (12.54)$$

Thus, we have proved the desired formula (12.48) for all values of L . This completes the proof of the lemma. \square

13 Dedekind sums.

The main result of this section will be used in §14. Let define the function $\varrho(x)$ in a following way $\varrho(x) = \frac{1}{2} - \{x\}$ for $x \in \mathbb{R}/\mathbb{Z}$ and $\varrho(x) = 0$ for $x \in \mathbb{Z}$. This section is devoted to the estimates of generalized Dedekind sums of the following form

$$\sum_{0 < n \leq P} \varrho(y_1 \frac{n}{P} + \frac{1}{R}) \varrho(y_2 \frac{n}{P} + \frac{1}{R}). \quad (13.1)$$

The proof of the following statement is based on the Knuth's article [8].

Lemma 13.1. *let $(y_1, y_2) = 1$, P be a natural number, R be a real number and $y_1, y_2 < \frac{R}{10}$. Then*

$$\sum_{0 < n \leq P} \mathbf{1}_{\{\|y_1 \frac{n}{P}\| < \frac{1}{R}\}} \mathbf{1}_{\{\|y_2 \frac{n}{P}\| < \frac{1}{R}\}} \leq 4 \frac{(y_1, P) + (y_2, P)}{R} + \frac{2P}{R} \min\{\frac{1}{y_1}, \frac{1}{y_2}\} + \frac{4P}{R^2} + O(1). \quad (13.2)$$

Proof. For any fixed real number a in the interval $0 < a < \frac{1}{2}$ one has

$$\begin{aligned} \mathbf{1}_{\{\|x\| < a\}} &= 2a + \rho(x+a) - \rho(x-a), \text{ если } x \neq \pm a \\ \mathbf{1}_{\{\|x\| < a\}} &< 2a + \rho(x+a) - \rho(x-a), \text{ если } x = \pm a, \end{aligned}$$

and, therefore,

$$\mathbf{1}_{\{\|y_1 \frac{n}{P}\| < \frac{1}{R}\}} \mathbf{1}_{\{\|y_2 \frac{n}{P}\| < \frac{1}{R}\}} \leq \prod_{i=1,2} \left(\frac{2}{R} + \varrho(y_i \frac{n}{P} + \frac{1}{R}) - \varrho(y_i \frac{n}{P} - \frac{1}{R}) \right). \quad (13.3)$$

Let $\varrho(y_i \frac{n}{P} \pm \frac{1}{R}) = \varrho_i^\pm$, then we obtain

$$\begin{aligned} \sum_{0 < n \leq P} \mathbf{1}_{\{\|y_1 \frac{n}{P}\| < \frac{1}{R}\}} \mathbf{1}_{\{\|y_2 \frac{n}{P}\| < \frac{1}{R}\}} &\leq \frac{4P}{R^2} + \frac{2}{R} \sum_{0 < n \leq P} (\varrho_1^+ - \varrho_1^- + \varrho_2^+ - \varrho_2^-) + \\ &+ \sum_{0 < n \leq P} (\varrho_1^+ \varrho_2^+ + \varrho_1^- \varrho_2^- - \varrho_1^- \varrho_2^+ - \varrho_1^+ \varrho_2^-) = \frac{4P}{R^2} + \Sigma_1 + \Sigma_2. \end{aligned} \quad (13.4)$$

To evaluate Σ_1 we need the following well known result. For $(p, q) = 1$ and for any real x one has

$$\sum_{n=1}^q \varrho\left(\frac{p}{q}n + x\right) = \varrho(qx) \quad (13.5)$$

see [10, стр.170, лемма 483]. Let estimate one of the four summands in the sum Σ_1 . Using (13.5), we obtain

$$\sum_{0 < n \leq P} \varrho\left(y_1 \frac{n}{P} + \frac{1}{R}\right) = (y_1, P) \varrho\left(\frac{P}{(y_1, P)} \frac{1}{R}\right). \quad (13.6)$$

Substituting (13.6) into the definition of Σ_1 , we obtain

$$|\Sigma_1| \leq 4 \frac{(y_1, P) + (y_2, P)}{R}. \quad (13.7)$$

To obtain an estimate on the sum Σ_2 we transform in the same manner each of the four summands of it. Consider the first summand

$$\sum_{0 < n \leq P} \varrho\left(y_1 \frac{n}{P} + \frac{1}{R}\right) \varrho\left(y_2 \frac{n}{P} + \frac{1}{R}\right).$$

Step 1 We denote

$$\delta_1 = (y_1, P), \delta_2 = (y_2, P), y_3 = \frac{y_1}{\delta_1}, y_4 = \frac{y_2}{\delta_2}, P_1 = \frac{P}{\delta_1}, P_2 = \frac{P}{\delta_1 \delta_2}$$

and prove that

$$\sum_{0 < n \leq P} \varrho\left(y_1 \frac{n}{P} + \frac{1}{R}\right) \varrho\left(y_2 \frac{n}{P} + \frac{1}{R}\right) = \sum_{0 < n \leq P_2} \varrho\left(y_3 \frac{n}{P_2} + \frac{\delta_2}{R}\right) \varrho\left(y_4 \frac{n}{P_2} + \frac{\delta_1}{R}\right). \quad (13.8)$$

Actually, the change of variables $n = (k-1)P_1 + m$ leads to

$$\begin{aligned} \sum_{0 < n \leq P} \varrho\left(y_1 \frac{n}{P} + \frac{1}{R}\right) \varrho\left(y_2 \frac{n}{P} + \frac{1}{R}\right) &= \sum_{1 \leq k \leq \delta_1} \sum_{1 \leq m \leq P_1} \varrho\left(y_1 \frac{k-1}{\delta_1} + \frac{y_1}{\delta_1} \frac{m}{P_1} + \frac{1}{R}\right) \varrho\left(y_2 \frac{k-1}{\delta_1} + \frac{m y_2}{P} + \frac{1}{R}\right) = \\ &= \sum_{1 \leq m \leq P_1} \varrho\left(y_3 \frac{m}{P_1} + \frac{1}{R}\right) \sum_{1 \leq k \leq \delta_1} \varrho\left(y_2 \frac{k}{\delta_1} + \frac{y_2(m - P_1)}{P} + \frac{1}{R}\right). \end{aligned}$$

Applying formula (13.5) to the sum over k we obtain

$$\sum_{1 \leq k \leq \delta_1} \varrho\left(y_2 \frac{k}{\delta_1} + \frac{y_2(m - P_1)}{P} + \frac{1}{R}\right) = \varrho\left(\frac{y_2(m - P_1)}{P_1} + \frac{\delta_1}{R}\right) = \varrho\left(\frac{y_2 m}{P_1} + \frac{\delta_1}{R}\right),$$

and, hence,

$$\sum_{0 < n \leq P} \varrho\left(y_1 \frac{n}{P} + \frac{1}{R}\right) \varrho\left(y_2 \frac{n}{P} + \frac{1}{R}\right) = \sum_{1 \leq m \leq P_1} \varrho\left(y_3 \frac{m}{P_1} + \frac{1}{R}\right) \varrho\left(\frac{y_2 m}{P_1} + \frac{\delta_1}{R}\right). \quad (13.9)$$

We note that $(y_3, P_1) = 1$, and as $(y_1, y_2) = 1$ so $(y_2, P_1) = (y_2, P) = \delta_2$. Changing the variables $m = (k-1)P_2 + n$ and repeating the proof of (13.9), we obtain

$$\sum_{1 \leq m \leq P_1} \varrho\left(y_3 \frac{m}{P_1} + \frac{1}{R}\right) \varrho\left(\frac{y_2 m}{P_1} + \frac{\delta_1}{R}\right) = \sum_{0 < n \leq P_2} \varrho\left(y_3 \frac{n}{P_2} + \frac{\delta_2}{R}\right) \varrho\left(y_4 \frac{n}{P_2} + \frac{\delta_1}{R}\right). \quad (13.10)$$

The relation (13.8) follows from (13.9) and (13.10).

Step 2 We find x, y such that

$$\frac{\delta_2}{R} = \frac{x}{P_2} + \theta_2, \quad \frac{\delta_1}{R} = \frac{y}{P_2} + \theta_1, \quad 0 \leq \theta_1, \theta_2 < \frac{1}{P_2}, \quad (13.11)$$

and prove that

$$\sum_{0 < n \leq P_2} \varrho\left(\frac{ny_3}{P_2} + \frac{\delta_2}{R}\right) \varrho\left(\frac{ny_4}{P_2} + \frac{\delta_1}{R}\right) = \sum_{0 < n \leq P_2} \varrho\left(\frac{ny_3}{P_2} + \frac{x}{P_2}\right) \varrho\left(\frac{ny_4}{P_2} + \frac{y}{P_2}\right) + O(1). \quad (13.12)$$

Actually, in view of

$$\varrho\left(\frac{a}{P_2}\right) - \varrho\left(\frac{a}{P_2} + \theta\right) = \begin{cases} -\varrho(\theta), & \text{if } a \equiv 0 \pmod{P_2}; \\ \theta, & \text{else,} \end{cases}$$

we have

$$\begin{aligned} \sum_{0 < n \leq P_2} \varrho\left(\frac{ny_3}{P_2} + \frac{x}{P_2}\right) \varrho\left(\frac{ny_4}{P_2} + \frac{\delta_1}{R}\right) - \sum_{0 < n \leq P_2} \varrho\left(\frac{ny_3}{P_2} + \frac{\delta_2}{R}\right) \varrho\left(\frac{ny_4}{P_2} + \frac{\delta_1}{R}\right) \\ = \theta_2 \sum_{0 < n \leq P_2} \varrho\left(\frac{ny_4}{P_2} + \frac{\delta_1}{R}\right) + O(1). \end{aligned} \quad (13.13)$$

Evaluating by the formula (13.5) the sum in the right side of (13.13) we obtain

$$\sum_{0 < n \leq P_2} \varrho\left(\frac{ny_3}{P_2} + \frac{\delta_2}{R}\right) \varrho\left(\frac{ny_4}{P_2} + \frac{\delta_1}{R}\right) = \sum_{0 < n \leq P_2} \varrho\left(\frac{ny_3}{P_2} + \frac{x}{P_2}\right) \varrho\left(\frac{ny_4}{P_2} + \frac{\delta_1}{R}\right) + O(1). \quad (13.14)$$

The similar transformations of the right side of (13.14) lead to the right side of (13.12).

Step 3 We make the change of variables $m \equiv ny_3 + x \pmod{P_2}$ in the right side of (13.12). Since $(y_3, P_2) = 1$, then y_3^{-1} is defined modulo P_2 . Hence, $n \equiv my_3^{-1} - xy_3^{-1} \pmod{P_2}$ and

$$\sum_{0 < n \leq P_2} \varrho\left(\frac{ny_3}{P_2} + \frac{x}{P_2}\right) \varrho\left(\frac{ny_4}{P_2} + \frac{y}{P_2}\right) = \sum_{0 < m \leq P_2} \varrho\left(\frac{m}{P_2}\right) \varrho\left(\frac{cm + z}{P_2}\right), \quad (13.15)$$

where $c \equiv y_4 y_3^{-1} \pmod{P_2}$, $z = y - cx$.

Let

$$V(z) = \sum_{0 < m \leq P_2} \varrho\left(\frac{m}{P_2}\right) \varrho\left(\frac{cm + z}{P_2}\right),$$

then we have proved that

$$\sum_{0 < n \leq P} \varrho\left(y_1 \frac{n}{P} + \frac{1}{R}\right) \varrho\left(y_2 \frac{n}{P} + \frac{1}{R}\right) = V(y - cx) + O(1),$$

where $c \equiv y_4 y_3^{-1} \pmod{P_2}$. Therefore,

$$\sum_{0 < n \leq P} (\varrho_1^+ \varrho_2^+ + \varrho_1^- \varrho_2^- - \varrho_1^- \varrho_2^+ - \varrho_1^+ \varrho_2^-) = V(y - cx) + V(-y + cx) - V(y + cx) - V(-y - cx) + O(1).$$

It is proved in [8, лемма 2] that $V(z) = V(-z)$, thus

$$\sum_{0 < n \leq P} (\varrho_1^+ \varrho_2^+ + \varrho_1^- \varrho_2^- - \varrho_1^- \varrho_2^+ - \varrho_1^+ \varrho_2^-) = 2V(cx - y) - 2V(cx + y) + O(1). \quad (13.16)$$

We note that by the symmetry of the left side of (13.15) with respect to x and y , one can assume, without loss of generality, that $x \geq y$, $1 \leq c < P_2$. If one of the numbers x, y is equal to zero then we obtain by (13.16) that

$$\sum_{0 < n \leq P} (\varrho_1^+ \varrho_2^+ + \varrho_1^- \varrho_2^- - \varrho_1^- \varrho_2^+ - \varrho_1^+ \varrho_2^-) = O(1).$$

So, further we assume $x \geq y \geq 1$.

Step 4 For $z > 0$ we prove that

$$V(z) = V(0) - \sum_{j=1}^z \varrho\left(\frac{c^{-1}j}{P_2}\right) + O(1), \quad (13.17)$$

where $cc^{-1} \equiv 1 \pmod{P_2}$. Since for an integer a one has

$$\varrho\left(\frac{a}{P_2}\right) - \varrho\left(\frac{a-1}{P_2}\right) = \begin{cases} \frac{1}{2} - \frac{1}{P_2}, & \text{if } a \equiv 0 \pmod{P_2}; \\ \frac{1}{2} - \frac{1}{P_2}, & \text{if } a \equiv 1 \pmod{P_2}; \\ -\frac{1}{P_2}, & \text{else,} \end{cases}$$

we have

$$\begin{aligned} V(z) - V(z-1) &= \sum_{0 < m \leq P_2} \varrho\left(\frac{m}{P_2}\right) \left(\varrho\left(\frac{cm+z}{P_2}\right) - \varrho\left(\frac{cm+z-1}{P_2}\right) \right) = \\ &= -\frac{1}{P_2} \sum_{0 < m \leq P_2} \varrho\left(\frac{m}{P_2}\right) + \frac{1}{2} \left(\varrho\left(\frac{m_1}{P_2}\right) + \varrho\left(\frac{m_2}{P_2}\right) \right), \end{aligned} \quad (13.18)$$

where $cm_1 + z \equiv 0 \pmod{P_2}$ и $cm_2 + z - 1 \equiv 0 \pmod{P_2}$. Using the formula (13.5), we obtain

$$V(z) - V(z-1) = -\frac{1}{2} \left(\varrho\left(\frac{c^{-1}z}{P_2}\right) + \varrho\left(\frac{c^{-1}(z-1)}{P_2}\right) \right). \quad (13.19)$$

In a similar manner we obtain

$$\begin{aligned} V(z-1) - V(z-2) &= -\frac{1}{2} \left(\varrho\left(\frac{c^{-1}(z-1)}{P_2}\right) + \varrho\left(\frac{c^{-1}(z-2)}{P_2}\right) \right), \\ &\vdots \\ V(1) - V(0) &= -\frac{1}{2} \left(\varrho\left(\frac{c^{-1}}{P_2}\right) + 0 \right). \end{aligned} \quad (13.20)$$

Adding up (13.19) and (13.20), we obtain

$$V(z) - V(0) = -\sum_{j=1}^z \varrho\left(\frac{c^{-1}j}{P_2}\right) + O(1).$$

Thus the relation (13.17) is proved.

Let $h = c^{-1} \equiv y_4^{-1} y_3 \pmod{P_2}$, $1 \leq h < P_2$. Substituting (13.17) into (13.16), we obtain

$$\sum_{0 < n \leq P} (\varrho_1^+ \varrho_2^+ + \varrho_1^- \varrho_2^- - \varrho_1^- \varrho_2^+ - \varrho_1^+ \varrho_2^-) = 2 \sum_{cx-y \leq j \leq cx+y} \varrho\left(\frac{hj}{P_2}\right) + O(1). \quad (13.21)$$

We transform the sum in the right side of (13.21). The change of variables $j = cx + n$ leads to

$$\begin{aligned} \sum_{cx-y \leq j \leq cx+y} \varrho\left(\frac{hj}{P_2}\right) &= \sum_{-y \leq n \leq y} \varrho\left(\frac{hn+x}{P_2}\right) = \\ &= \sum_{0 < n \leq y} \left(\varrho\left(\frac{hn+x}{P_2}\right) - \varrho\left(\frac{hn-x}{P_2}\right) \right) + O(1) \leq \\ &\leq \sum_{0 < n \leq y} \left(\frac{2x}{P_2} + \varrho\left(\frac{hn+x}{P_2}\right) - \varrho\left(\frac{hn-x}{P_2}\right) \right) + O(1) = \sum_{0 < n \leq y} \mathbf{1}_{\{\|\frac{hn}{P_2}\| \leq \frac{x}{P_2}\}} + O(1). \end{aligned}$$

Hence,

$$\sum_{0 < n \leq P} (\varrho_1^+ \varrho_2^+ + \varrho_1^- \varrho_2^- - \varrho_1^- \varrho_2^+ - \varrho_1^+ \varrho_2^-) \leq 2 \sum_{0 < n \leq y} \mathbf{1}_{\{\|\frac{hn}{P_2}\| \leq \frac{x}{P_2}\}} + O(1). \quad (13.22)$$

By (13.11), taking into account the inequality $x \geq y \geq 1$, we obtain

$$\frac{x}{P_2} \leq \frac{\delta_2}{R} < \frac{x+1}{P_2} \leq \frac{2x}{P_2}, \quad \frac{y}{P_2} \leq \frac{\delta_1}{R} < \frac{y+1}{P_2} \leq \frac{2y}{P_2}, \quad (13.23)$$

and, therefore,

$$\frac{\delta_2}{2R} < \frac{x}{P_2} \leq \frac{\delta_2}{R}, \quad \frac{\delta_1}{2R} < \frac{y}{P_2} \leq \frac{\delta_1}{R}. \quad (13.24)$$

In particular, $\frac{\delta_2}{R} \geq \frac{1}{P_2}$. Then one has $P_2 \geq \frac{R}{\delta_2} > \frac{y_2}{\delta_2} = y_4$ and, in the same manner, $P_2 > y_3$. Further, by the definition one has $hy_4 \equiv y_3 \pmod{P_2}$. Then in view of $1 \leq h < P_2$, $0 < y_3, y_4 < P_2$ there exists k such that

$$hy_4 = y_3 + kP_2$$

and $0 \leq k < y_4$. We note that $(y_3, y_4) = 1$ implies that $(k, y_4) = 1$. Hence,

$$\frac{hn}{P_2} = \frac{n}{P_2} \frac{y_3 + kP_2}{y_4} = \frac{nk + \frac{ny_3}{P_2}}{y_4}. \quad (13.25)$$

We consider two cases depending on the value of k .

1. If $k = 0$, then $hy_4 = y_3$. So, because of $(y_4, y_3) = 1$, we obtain $y_4 = 1$, that is, $y_2 = \delta_2$. Then it follows from (13.25) that

$$\sum_{0 < n \leq y} \mathbf{1}_{\{\|\frac{hn}{P_2}\| \leq \frac{x}{P_2}\}} = \sum_{0 < n \leq y} \mathbf{1}_{\{\|\frac{ny_3}{P_2}\| \leq \frac{x}{P_2}\}}. \quad (13.26)$$

Using the trivial estimate of the right side of (13.26) and the bound (13.24) we obtain

$$\sum_{0 < n \leq y} \mathbf{1}_{\{\|\frac{hn}{P_2}\| \leq \frac{x}{P_2}\}} \leq y \leq \frac{\delta_1 P_2}{R} = \frac{P}{R\delta_2} = \frac{P}{Ry_2}. \quad (13.27)$$

On the other hand, since $yy_3 = y \frac{y_1}{\delta_1} \leq \frac{\delta_1 P_2}{R} \frac{y_1}{\delta_1} < \frac{P_2}{2}$, one has

$$\sum_{0 < n \leq y} \mathbf{1}_{\{\|\frac{ny_3}{P_2}\| \leq \frac{x}{P_2}\}} \leq \frac{x}{y_3} \leq \frac{\delta_2 P_2}{R} \frac{\delta_1}{y_1} = \frac{P}{Ry_1}. \quad (13.28)$$

Hence, by (13.28) and (13.27) we obtain

$$\sum_{0 < n \leq y} \mathbf{1}_{\{\|\frac{hn}{P_2}\| \leq \frac{x}{P_2}\}} \leq \frac{P}{R} \min\left\{\frac{1}{y_1}, \frac{1}{y_2}\right\}. \quad (13.29)$$

2. If $k \neq 0$, then it is necessary to study the inequality

$$\left\| \frac{nk + \frac{ny_3}{P_2}}{y_4} \right\| \leq \frac{x}{P_2}. \quad (13.30)$$

Because

$$\frac{x}{P_2} = \frac{1}{y_4} \frac{xy_4}{P_2} \leq \frac{1}{y_4} \frac{\delta_2 y_4}{R} = \frac{1}{y_4} \frac{y_2}{R} \leq \frac{1}{10y_4},$$

$$\frac{ny_3}{P_2} \leq \frac{yy_3}{P_2} \leq \frac{\delta_1 y_1}{R \delta_1} \leq \frac{y_1}{R} \leq \frac{1}{10}$$

the inequality (13.30) holds only if $y_4 | nk$. Since $(k, y_4) = 1$, one has $y_4 | n$. Therefore,

$$\sum_{0 < n \leq y} \mathbf{1}_{\{\|\frac{hn}{P_2}\| \leq \frac{x}{P_2}\}} \leq \sum_{0 < n \leq \frac{y}{y_4}} \mathbf{1}_{\{\|\frac{ny_3}{P_2}\| \leq \frac{x}{P_2}\}} \leq \min\left\{\frac{y}{y_4}, \frac{x}{y_3}\right\} \leq \frac{P}{R} \min\left\{\frac{1}{y_1}, \frac{1}{y_2}\right\}. \quad (13.31)$$

Substituting (13.31) or (13.29) into (13.22), we obtain

$$\sum_{0 < n \leq P} (\varrho_1^+ \varrho_2^+ + \varrho_1^- \varrho_2^- - \varrho_1^- \varrho_2^+ - \varrho_1^+ \varrho_2^-) \leq \frac{2P}{R} \min\left\{\frac{1}{y_1}, \frac{1}{y_2}\right\} + O(1). \quad (13.32)$$

Substituting (13.7) and (13.32) into (13.4), we obtain (13.2). This completes the proof of the lemma. \square

14 The case $\mu = 3$.

Let Q_1, Q, β be given. For any q in $Q_1 \leq q \leq Q$ we define by any means the number a_q , such that $(a_q, q) = 1$, $0 \leq a_q \leq q$. Let denote

$$Z^* = \left\{ \theta = \frac{a_q}{q} + \beta \mid Q_1 \leq q \leq Q \right\}. \quad (14.1)$$

Lemma 14.1. *Let $\mu = 3$, $\lambda = 2$, then for any $Z \subseteq Z^*$ the following bound on the cardinality of the set $\mathfrak{M}(g_2)$ holds*

$$|\mathfrak{M}(g_2)| \leq |Z| |\Omega^{(3)}| \left(1 + \frac{2H_3}{Q_1}\right) \left(1 + \frac{146A^2N}{KQ_1M^{(1)}\|g_2\|}\right). \quad (14.2)$$

Proof. We use the partition of the ensemble Ω_N given by the formula (12.6) (that is, by Theorem 11.5). For $\mu = 3$ the equation (12.44) becomes the congruence

$$\left(g_2(g_3^{(1)} \frac{a^{(1)}}{q^{(1)}} - g_3^{(2)} \frac{a^{(2)}}{q^{(2)}}) \right)_{1,2} \equiv 0 \pmod{1}, \quad (14.3)$$

where the subscripts "1,2" mean that the congruence (14.3) holds for both coordinates of the vector. Let $\mathbf{q} = [q^{(1)}, q^{(2)}]$, then (14.3) can be written as

$$\left(g_2(g_3^{(1)} \frac{a^{(1)}q^{(2)}}{(q^{(1)}, q^{(2)})} - g_3^{(2)} \frac{a^{(2)}q^{(1)}}{(q^{(1)}, q^{(2)})}) \right)_{1,2} \equiv 0 \pmod{\mathbf{q}}. \quad (14.4)$$

Since $\det g_2 = 1$, the congruence (14.4) can be simplified to

$$\left(g_3^{(1)} \frac{a^{(1)}q^{(2)}}{(q^{(1)}, q^{(2)})} - g_3^{(2)} \frac{a^{(2)}q^{(1)}}{(q^{(1)}, q^{(2)})} \right)_{1,2} \equiv 0 \pmod{\mathbf{q}}. \quad (14.5)$$

By setting $g_3^{(1)} = (x_1, x_2)^t$, $g_3^{(2)} = (y_1, y_2)^t$ in (14.5) we obtain the congruence

$$x_1 \frac{a^{(1)}q^{(2)}}{(q^{(1)}, q^{(2)})} \equiv y_1 \frac{a^{(2)}q^{(1)}}{(q^{(1)}, q^{(2)})} \pmod{\mathbf{q}} \quad (14.6)$$

and, the same one for x_2, y_2 . But $(a^{(1)}, \frac{q^{(1)}}{(q^{(1)}, q^{(2)})}) \leq (a^{(1)}, q^{(1)}) = 1$ and, therefore,

$$x_1 \equiv 0 \pmod{\frac{q^{(1)}}{(q^{(1)}, q^{(2)})}}, \quad x_2 \equiv 0 \pmod{\frac{q^{(1)}}{(q^{(1)}, q^{(2)})}}$$

and the same for y_1, y_2 . At the same time $(x_1, x_2) = (y_1, y_2) = 1$ as the component of the vectors $g_3^{(1)}, g_3^{(2)}$, thus

$$q^{(1)} = (q^{(1)}, q^{(2)}) = q^{(2)} = \mathbf{q}. \quad (14.7)$$

Let fix \mathbf{q} , for which there are $|Z|$ choices, this gives the first factor in (14.2). Then it follows from the conditions on the set Z that $a^{(1)} = a^{(2)}$ and the congruence (14.5) can be simplified to

$$(g_3^{(1)} - g_3^{(2)})_{1,2} \equiv 0 \pmod{\mathbf{q}}. \quad (14.8)$$

We choose and fix the vector $g_3^{(2)}$, for which there are $|\Omega^{(3)}|$ choices. This gives the second factor in (14.2). In view of Theorem 11.5 we have $|g_3^{(1)} - g_3^{(2)}|_{1,2} \leq H_3$. Putting $z = \{z_1, z_2\} = g_3^{(1)} - g_3^{(2)}$ and using (14.8) we obtain that there are at most $\left(1 + \frac{2H_3}{\mathbf{q}}\right)$ choices for z_1 . This gives the third factor in (14.2). Finally, putting $\eta = \frac{73A^2N}{\bar{K}M^{(1)}}$ and $g_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $d = \|g_2\|$, we write the inequality (12.43) in the form $|g_2(g_3^{(1)} - g_3^{(2)})|_{1,2} \leq \eta$, whence

$$\frac{-\eta - cz_1}{d} \leq z_2 \leq \frac{\eta - cz_1}{d}.$$

Therefore, taking into account the congruence (14.8), we obtain that z_2 takes at most

$$\frac{2\eta}{\mathbf{q}\|g_2\|} + 1 = \left(1 + \frac{146A^2N}{\mathbf{q}\bar{K}M^{(1)}\|g_2\|}\right)$$

values. This gives the third factor in (14.2). This completes the proof of the lemma. \square

Let

$$Q_0 = \max \left\{ \exp \left(\frac{10^5 A^4}{\epsilon_0^2} \right), \exp(\epsilon_0^{-5}) \right\}. \quad (14.9)$$

Lemma 14.2. *If $\overline{K}^2 Q^3 \leq \frac{N^{1-\epsilon_0}}{12000 A^4}$, $\overline{K} Q \geq Q_0$, then the following bound holds*

$$\sum_{\theta \in P_{Q_1, Q}^{(\beta)}} |S_N(\theta)|^2 \ll |\Omega_N|^2 \overline{K}^{12\epsilon_0} Q^{20\epsilon_0} \frac{\overline{K}^{4(1-\delta)} Q^{6(1-\delta)+1}}{\overline{K} Q_1^2} \quad (14.10)$$

Proof. We use the partition of the ensemble Ω_N given by the formula (12.6) (that is, by Theorem 11.5). Let $Z \subseteq Z^*$ be any subset. We put $\mu = 3, \lambda = 2$ and

$$M^{(1)} = 76 A^2 \overline{K} Q^2, \quad M^{(3)} = 76 A^2 \overline{K} Q. \quad (14.11)$$

Then the condition (12.45) holds. It follows from the statement of the lemma that inequalities (11.41) and (11.42) hold. So all conditions of Lemma 12.4 hold and therefore the estimate (12.46) is valid. Since $H_3 \geq M^{(3)} \geq Q_1$ the third factor in (14.2) can be replaced by $\frac{3H_3}{Q_1}$. Let consider the fourth factor in (14.2). Using the lower bound of (11.55), we obtain:

$$1 + \frac{146 A^2 N}{\overline{K} Q_1 M^{(1)} \|g_2\|} \leq 1 + \frac{146 A^2 N}{\overline{K} Q_1 M^{(1)}} \frac{3 \cdot 10^4 A^5 (M^{(1)} M^{(3)})^{1+2\epsilon_0}}{N} \ll \left(\overline{K}^2 Q^3 \right)^{2\epsilon_0} \frac{Q}{Q_1}, \quad (14.12)$$

where we have used (14.11). Substituting the result of the simplification into (14.2), we obtain

$$|\mathfrak{M}(g_2)| \ll |Z| |\Omega^{(3)}| \left(\overline{K}^2 Q^3 \right)^{2\epsilon_0} \frac{H_3 Q}{Q_1^2}. \quad (14.13)$$

Substituting $H_3(M^{(3)})$ from (12.9), we obtain

$$|\mathfrak{M}(g_2)| \ll |Z| |\Omega^{(3)}| \left(\overline{K}^3 Q^4 \right)^{2\epsilon_0} \frac{\overline{K} Q^2}{Q_1^2}. \quad (14.14)$$

Substituting the obtained bound on $|\mathfrak{M}(g_2)|$ into (12.46), we have

$$\sigma_{N,Z} \ll H_1 |\Omega^{(1)}|^{1/2} |\Omega^{(2)}| |\Omega^{(3)}|^{1/2} |Z|^{1/2} \left(\overline{K}^3 Q^4 \right)^{\epsilon_0} \frac{\overline{K}^{1/2} Q}{Q_1}. \quad (14.15)$$

Using the estimate (11.63), we obtain

$$\sigma_{N,Z} \ll |\Omega_N| \frac{(M^{(1)})^{1+2\epsilon_0}}{(M^{(1)} M^{(3)})^{\delta-\epsilon_0/2}} |Z|^{1/2} \left(\overline{K}^3 Q^4 \right)^{\epsilon_0} \frac{\overline{K}^{1/2} Q}{Q_1}.$$

Substituting $M^{(1)}, M^{(3)}$ into (14.11), we have

$$\sigma_{N,Z} \ll |Z|^{1/2} |\Omega_N| \overline{K}^{6\epsilon_0} Q^{9.5\epsilon_0} \frac{\overline{K}^{3/2-2\delta} Q^{3(1-\delta)}}{Q_1}. \quad (14.16)$$

Applying Lemma 12.5 with $W = Z^*, C_2 = 0$, we deduce from (14.16) that

$$\sum_{\theta \in Z^*} |S_N(\theta)|^2 \ll |\Omega_N|^2 \overline{K}^{12\epsilon_0} Q^{20\epsilon_0} \frac{\overline{K}^{4(1-\delta)} Q^{6(1-\delta)}}{\overline{K} Q_1^2}. \quad (14.17)$$

Using the trivial bound

$$\sum_{\theta \in P_{Q_1, Q}^{(\beta)}} |S_N(\theta)|^2 \leq Q \sum_{Q_1 \leq q \leq Q} \max_{1 \leq a \leq q, (a, q)=1} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 = Q \sum_{\theta \in Z^*} |S_N(\theta)|^2, \quad (14.18)$$

where as a_q we have chosen numerators for which the maximum is achieved, and substituting the obtained bound (14.17) into (14.18), we obtained the desired estimate. This completes the proof of the lemma. \square

Lemma 14.3. *If $\overline{K}^2 q^2 \leq \frac{N^{1-\epsilon_0}}{12000A^4}$, $\overline{K}q \geq Q_0$, then for $\theta = \frac{a}{q} + \frac{K}{N}$ the following bound holds*

$$|S_N(\theta)| \ll |\Omega_N| (\overline{K}q)^{6\epsilon_0} \frac{(\overline{K}q)^{2(1-\delta)}}{\overline{K}^{1/2} q}. \quad (14.19)$$

Proof. Let $Z = \{\theta\}$. We use the partition of the ensemble Ω_N given by the formula (12.6) (that is, by Theorem 11.5). We put $\mu = 3, \lambda = 2$ and

$$M^{(1)} = 76A^2 \overline{K}q, \quad M^{(3)} = 76A^2 \overline{K}q. \quad (14.20)$$

Then the condition (12.45) holds. It follows from the statement of the lemma that inequalities (11.41) and (11.42) hold. So all conditions of Lemma 12.4 hold. Using Lemma 14.1 with $|Z| = 1$ and making all transformations in the same way as in Lemma 14.2, we obtain

$$|S_N(\theta)| = \sigma_{N, Z} \ll |\Omega_N| (\overline{K}q)^{6\epsilon_0} \frac{(\overline{K}q)^{2(1-\delta)}}{\overline{K}^{1/2} q}. \quad (14.21)$$

This completes the proof of the lemma. \square

Lemma 14.4. *If $\overline{K}q \geq Q_0$, then the following bound holds*

$$|S_N(\theta)| \ll |\Omega_N| \frac{(\overline{K}q)^{2\epsilon_0} N^{1-\delta+\epsilon_0}}{\overline{K}q}. \quad (14.22)$$

Proof. We use the partition of the ensemble Ω_N given by the formula (12.8) (that is, by Theorem 11.3). We put $\mu = 3, \lambda = 2, Z = \{\theta\}$ and

$$M^{(1)} = 76A^2 \overline{K}q. \quad (14.23)$$

Then the condition (12.45) holds and therefore the estimate (12.46) is valid. Since $\Omega^{(2)} = \{E\}$, the conditions (12.43) and (12.44) can be written as

$$g_3^{(1)} \equiv g_3^{(2)} \pmod{q}, \quad |g_3^{(1)} - g_3^{(2)}|_{1,2} \leq \frac{73A^2 N}{M^{(1)} \overline{K}}. \quad (14.24)$$

Thus we obtain

$$\begin{aligned} |\mathfrak{M}(g_\lambda)| &\leq \sum_{g_3^{(2)} \in \Omega^{(3)}} \sum_{g_3^{(1)} \in \mathbb{Z}^2} \mathbf{1}_{\{g_3^{(1)} \equiv g_3^{(2)} \pmod{q}, |g_3^{(1)} - g_3^{(2)}|_{1,2} \leq \frac{N}{\overline{K}^2 q}\}} \leq \\ &\leq |\Omega^{(3)}| \left(1 + \frac{N}{\overline{K}^2 q^2}\right)^2 \ll \frac{|\Omega^{(3)}| N^2}{\overline{K}^4 q^4}, \end{aligned} \quad (14.25)$$

as, in view of (12.26) we have $\overline{K}q \leq N^{1/2}$. Substituting (14.25) into (12.46), we obtain

$$|S_N(\theta)| \ll H_1 |\Omega^{(1)}|^{1/2} |\Omega^{(3)}|^{1/2} \frac{N}{\overline{K}^2 q^2} \ll |\Omega_N| \frac{(M^{(1)})^{1+2\epsilon_0}}{|\Omega_N|^{1/2}} \frac{N}{\overline{K}^2 q^2}. \quad (14.26)$$

Using (14.23) and the lower bound of (11.38), we have

$$|S_N(\theta)| \ll |\Omega_N| \frac{(\overline{K}q)^{1+2\epsilon_0} N^{1-\delta+\epsilon_0}}{\overline{K}^2 q^2}.$$

This completes the proof of the lemma. \square

Lemma 14.5. *Let the following inequalities hold $N^{\epsilon_0/2} \leq Q^{1/2} \leq Q_1 \leq Q$, $\overline{K}Q \leq N^\alpha$, and $\frac{1}{4} < \alpha \leq \frac{1}{2} + \epsilon_0$. Then the bound*

$$\sum_{\theta \in P_{Q_1, Q}^{(\beta)}} |S_N(\theta)| \ll |\Omega_N| \left(N^{1/2+\alpha/2-\delta+3\epsilon_0} Q + N^{1-\delta+3\epsilon_0} \frac{Q^{1/2}}{\overline{K}} \right) \quad (14.27)$$

holds. Moreover, for any $Z \subseteq P_{Q_1, Q}^{(\beta)}$ the following bound holds

$$\begin{aligned} \sum_{\theta \in Z} |S_N(\theta)| &\ll |\Omega_N| |Z|^{1/2} \left(\frac{N^{1-\delta+2\epsilon_0}}{\overline{K} Q_1^{1/2}} + N^{1-\frac{3-\alpha}{2}\delta+4,5\epsilon_0} + N^{\frac{3+\alpha}{4}-\frac{3-\alpha}{2}\delta+4,5\epsilon_0} Q^{1/2} \right) + \\ &+ |\Omega_N| N^{\frac{1+\alpha}{2}-\delta+2,5\epsilon_0} Q. \end{aligned} \quad (14.28)$$

Proof. The beginning of both proofs is the same. We use the partition of the ensemble Ω_N given by the formula (12.8) (that is, by Theorem 11.3). We put $\mu = 3, \lambda = 2, Z \subseteq P_{Q_1, Q}^{(\beta)}$ and

$$M^{(1)} = 150A^2 N^{1/2+\alpha/2}. \quad (14.29)$$

Then $M^{(1)} \geq 150A^2 N^\alpha \geq 150A^2 \overline{K}$ and so the conditions of Lemma 12.3 hold. Therefore, the inequality (12.34) holds. It follows from (12.31) and (12.32) that

$$\left\| g_3 \frac{a^{(1)}}{q^{(1)}} - g_3^{(2)} \frac{a^{(2)}}{q^{(2)}} \right\|_{1,2} \leq \frac{74A^2 \overline{K}}{M^{(1)}}, \quad (14.30)$$

$$|g_3 - g_3^{(2)}|_{1,2} \leq \min \left\{ \frac{73A^2 N}{M^{(1)}}, \frac{73A^2 N}{M^{(1)} \overline{K}} + \frac{N}{\overline{K}} \left\| g_3 \frac{a^{(1)}}{q^{(1)}} - g_3^{(2)} \frac{a^{(2)}}{q^{(2)}} \right\|_{1,2} \right\}, \quad (14.31)$$

Let $g_3^{(1)} = (x_1, x_2)^t, g_3^{(2)} = (y_1, y_2)^t$, then it follows from Theorem 11.3 that

$$\max\{y_1, y_2\} = y_2 \leq \frac{73A^2 N}{M^{(1)}}. \quad (14.32)$$

We denote

$$Y = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}, \quad \mathcal{Y} = \det(Y) = x_1 y_2 - y_1 x_2. \quad (14.33)$$

Then it follows from the triangle inequality that

$$\|\mathcal{Y} \frac{a^{(1)}}{q^{(1)}}\| \leq \|y_2(x_1 \frac{a^{(1)}}{q^{(1)}} - y_1 \frac{a^{(2)}}{q^{(2)}})\| + \|y_1(y_2 \frac{a^{(2)}}{q^{(2)}} - x_2 \frac{a^{(1)}}{q^{(1)}})\|. \quad (14.34)$$

Applying (14.30), (14.32) and (14.29) we similarly estimate both summands in the right side of (14.34)

$$\|\mathcal{Y} \frac{a^{(1)}}{q^{(1)}}\| \leq 2 \frac{74A^2 \overline{K}}{M^{(1)}} \frac{73A^2 N}{M^{(1)}} < \frac{1}{2Q} < \frac{1}{q^{(1)}}. \quad (14.35)$$

Hence, $\mathcal{Y} \equiv 0 \pmod{q^{(1)}}$ and similarly $\mathcal{Y} \equiv 0 \pmod{q^{(2)}}$. So one has $\mathcal{Y} \equiv 0 \pmod{\mathbf{q}}$, where $\mathbf{q} = [q^{(1)}, q^{(2)}]$ and so $Q_1 \leq \mathbf{q} \leq Q^2$.

To prove the inequality (14.27) we put $Z = P_{Q_1, Q}^{(\beta)}$ and use the bound (12.34). The set $\mathfrak{N}(g_2)$ can be represented as a union of the sets $\mathfrak{M}_1, \mathfrak{M}_2$. For the first set $\mathcal{Y} = 0$, for the second one $\mathcal{Y} \neq 0$.

1. To prove the estimate (14.27) we use the following bounds

$$|\mathfrak{M}_1| \ll_{\epsilon} Q^{2+\epsilon} |\Omega^{(3)}|, \quad (14.36)$$

$$|\mathfrak{M}_2| \ll_{\epsilon} |\Omega^{(3)}| \left(\left(\frac{73A^2 N}{M^{(1)} \overline{K}} \right)^2 + 1 \right) N^{\epsilon} Q, \quad (14.37)$$

which will be proved below in Lemma 14.6 and 14.7 respectively. Hence,

$$|\mathfrak{N}(g_2)|^{1/2} \ll_{\epsilon} |\Omega^{(3)}|^{1/2} \left(Q^{1+\epsilon} + \frac{73A^2 N^{1+\epsilon} Q^{1/2}}{M^{(1)} \overline{K}} + N^{\epsilon} Q^{1/2} \right). \quad (14.38)$$

Substituting (14.38) into (12.34), we obtain for $\epsilon = \epsilon_0$:

$$\sigma_{N, Z} \ll 10H_1 |\Omega^{(1)}|^{1/2} |\Omega^{(3)}|^{1/2} \left(Q^{1+\epsilon_0} + \frac{73A^2 N^{1+\epsilon_0} Q^{1/2}}{M^{(1)} \overline{K}} + N^{\epsilon_0} Q^{1/2} \right). \quad (14.39)$$

We note that it follows from the condition of the lemma that $Q^{1/2} N^{\epsilon_0/2} \leq Q$, so the last inequality in (14.39) can be omitted. Using the bounds (11.63) and (11.38), we obtain

$$|\Omega^{(1)}|^{1/2} |\Omega^{(3)}|^{1/2} = |\Omega_N| \frac{1}{|\Omega_N|^{1/2}} \leq |\Omega_N| \frac{1}{N^{\delta-\epsilon_0/2}}. \quad (14.40)$$

Substituting (14.40) into (14.39), we obtain

$$\sigma_{N, Z} \ll |\Omega_N| \frac{1}{N^{\delta-\epsilon_0/2}} \left((M^{(1)})^{1+2\epsilon_0} Q^{1+\epsilon_0} + \frac{N^{1+\epsilon_0} Q^{1/2} (M^{(1)})^{2\epsilon_0}}{\overline{K}} \right) \quad (14.41)$$

Substituting $M^{(1)}$ from (14.29), we obtain

$$\sigma_{N, Z} \ll |\Omega_N| \left(N^{1/2+\alpha/2-\delta+2.5\epsilon_0} Q + N^{1-\delta+3\epsilon_0} \frac{Q^{1/2}}{\overline{K}} \right). \quad (14.42)$$

Thus the bound (14.27) is proved.

2. To prove the estimate (14.28) we use the bound (14.36) and

$$|\mathfrak{M}_2| \ll_\epsilon |Z| \left(\left(\frac{73A^2N}{M^{(1)}} \right)^2 \frac{|\Omega^{(3)}|}{\bar{K}Q_1} + \left(\frac{73A^2N}{M^{(1)}} \right)^2 Q^\epsilon + \left(\frac{73A^2N}{M^{(1)}} \right) Q^{1+\epsilon} \right), \quad (14.43)$$

the last will be proved in Lemma 14.8. Hence,

$$|\mathfrak{N}(g_2)|^{1/2} \ll_\epsilon |Z|^{1/2} \left(\frac{N}{M^{(1)}} \frac{|\Omega^{(3)}|^{1/2}}{\bar{K}Q_1^{1/2}} + \frac{NQ^\epsilon}{M^{(1)}} + \left(\frac{N}{M^{(1)}} \right)^{1/2} Q^{1/2+\epsilon} \right) + Q^{1+\epsilon} |\Omega^{(3)}|^{1/2}. \quad (14.44)$$

Substituting (14.44) into (12.34), we obtain for $\epsilon = \epsilon_0$:

$$\begin{aligned} \sigma_{N,Z} \ll (M^{(1)})^{1+2\epsilon_0} |\Omega^{(1)}|^{1/2} |Z|^{1/2} & \left(\frac{N}{M^{(1)}} \frac{|\Omega^{(3)}|^{1/2}}{\bar{K}Q_1^{1/2}} + \frac{NQ^{\epsilon_0}}{M^{(1)}} + \left(\frac{N}{M^{(1)}} \right)^{1/2} Q^{1/2+\epsilon_0} \right) + \\ & + (M^{(1)})^{1+2\epsilon_0} |\Omega^{(1)}|^{1/2} Q^{1+\epsilon_0} |\Omega^{(3)}|^{1/2}. \end{aligned} \quad (14.45)$$

Using the bounds (11.63) and (11.38), we have

$$|\Omega^{(1)}|^{1/2} = |\Omega_N| \frac{|\Omega^{(1)}|^{1/2}}{|\Omega_N|} \leq |\Omega_N| \frac{(M^{(1)})^{\delta+2\epsilon_0}}{N^{2\delta-\epsilon_0}}. \quad (14.46)$$

Substituting (14.46) and (14.40) into (14.45), we obtain

$$\begin{aligned} \sigma_{N,Z} \ll |\Omega_N| |Z|^{1/2} & \left(\frac{N^{1-\delta+\epsilon_0/2} (M^{(1)})^{2\epsilon_0}}{\bar{K}Q_1^{1/2}} + N^{1-2\delta+\epsilon_0} (M^{(1)})^{\delta+4\epsilon_0} Q^{\epsilon_0} + \right. \\ & \left. + N^{1/2-2\delta+\epsilon_0} (M^{(1)})^{1/2+\delta+4\epsilon_0} Q^{1/2+\epsilon_0} \right) + |\Omega_N| \frac{(M^{(1)})^{1+2\epsilon_0}}{N^{\delta-\epsilon_0/2}} Q^{1+\epsilon}. \end{aligned}$$

Substituting $M^{(1)}$ from (14.29), we obtain

$$\begin{aligned} \sigma_{N,Z} \ll |\Omega_N| |Z|^{1/2} & \left(\frac{N^{1-\delta+2\epsilon_0}}{\bar{K}Q_1^{1/2}} + N^{1-\frac{3-\alpha}{2}\delta+4,5\epsilon_0} + N^{\frac{3+\alpha}{4}-\frac{3-\alpha}{2}\delta+4,5\epsilon_0} Q^{1/2} \right) + \\ & + |\Omega_N| N^{\frac{1+\alpha}{2}-\delta+2,5\epsilon_0} Q. \end{aligned}$$

Thus the bound (14.28) is proved.

This completes the proof of the lemma. \square

Lemma 14.6. *Under the hypotheses of Lemma 14.5 one has*

$$|\mathfrak{M}_1| \ll_\epsilon Q^{2+\epsilon} |\Omega^{(3)}|. \quad (14.47)$$

Proof. To simplify we denote

$$R = (M^{(1)})^{1+2\epsilon_0}, T = \frac{73A^2N}{M^{(1)}}.$$

We recall that $M^{(1)}$ is defined in (14.29). It follows from the condition $\mathcal{Y} = 0$ that $\frac{x_1}{x_2} = \frac{y_1}{y_2}$. Since

$$g_3^{(1)} = (x_1, x_2)^t, \quad g_3^{(2)} = (y_1, y_2)^t,$$

one has $(x_1, x_2) = 1, (y_1, y_2) = 1$ and, hence, $x_1 = y_1, x_2 = y_2$. In particular, $|\beta(g_3^{(1)} - g_3^{(2)})|_{1,2} = 0$. Then it follows from the inequality (12.29) that

$$\|y_1 \left(\frac{a^{(1)}}{q^{(1)}} - \frac{a^{(2)}}{q^{(2)}} \right)\| \leq \|y_1 (\theta^{(1)} - \theta^{(2)})\| + 0 \leq \frac{1}{H_1} < \frac{1}{R} \quad (14.48)$$

and, in the same way, $\|y_2 \left(\frac{a^{(1)}}{q^{(1)}} - \frac{a^{(2)}}{q^{(2)}} \right)\| \leq \frac{1}{H_1} < \frac{1}{R}$. Thus we obtain

$$|\mathfrak{M}_1| \leq \sum_{g_3^{(2)} \in \tilde{\Omega}^{(3)}} \sum_{Q_1 \leq q^{(1)}, q^{(2)} \leq Q} \sum_{0 \leq a^{(1)} \leq q^{(1)}}^* \sum_{0 \leq a^{(2)} \leq q^{(2)}}^* \mathbf{1}_{\{\|y_{1,2} \left(\frac{a^{(1)}}{q^{(1)}} - \frac{a^{(2)}}{q^{(2)}} \right)\| < \frac{1}{R}\}}. \quad (14.49)$$

We denote $P = [q^{(1)}, q^{(2)}]$ and write $\frac{a^{(1)}}{q^{(1)}} - \frac{a^{(2)}}{q^{(2)}} = \frac{n}{P}$. We note that

$$\#\{a^{(1)}, a^{(2)} \mid a^{(1)} \frac{P}{q^{(1)}} - a^{(2)} \frac{P}{q^{(2)}} = n, 1 \leq a^{(1)} \leq q^{(1)}, 1 \leq a^{(2)} \leq q^{(2)}\} \leq (q^{(1)}, q^{(2)}),$$

then (14.49) can be written as

$$\begin{aligned} |\mathfrak{M}_1| &\leq \sum_{g_3^{(2)} \in \tilde{\Omega}^{(3)}} \sum_{Q_1 \leq q^{(1)}, q^{(2)} \leq Q} (q^{(1)}, q^{(2)}) \sum_{|n| < P} \mathbf{1}_{\{\|y_1 \frac{n}{P}\| < \frac{1}{R}\}} \mathbf{1}_{\{\|y_2 \frac{n}{P}\| < \frac{1}{R}\}} \leq \\ &\leq 2 \sum_{g_3^{(2)} \in \tilde{\Omega}^{(3)}} \sum_{Q_1 \leq q^{(1)}, q^{(2)} \leq Q} (q^{(1)}, q^{(2)}) \sum_{0 < n \leq P} \mathbf{1}_{\{\|y_1 \frac{n}{P}\| < \frac{1}{R}\}} \mathbf{1}_{\{\|y_2 \frac{n}{P}\| < \frac{1}{R}\}}. \end{aligned} \quad (14.50)$$

Applying Lemma 13.1, we obtain

$$|\mathfrak{M}_1| \leq 8 \sum_{g_3^{(2)} \in \tilde{\Omega}^{(3)}} \sum_{Q_1 \leq q^{(1)}, q^{(2)} \leq Q} (q^{(1)}, q^{(2)}) \left(\frac{(y_1, P) + (y_2, P)}{R} + \frac{P}{R} \min\left\{\frac{1}{y_1}, \frac{1}{y_2}\right\} + \frac{P}{R^2} + O(1) \right). \quad (14.51)$$

Using (11.46), we have

$$\frac{P}{R} \min\left\{\frac{1}{y_1}, \frac{1}{y_2}\right\} = \frac{P}{R y_2} \leq \frac{P}{R} \frac{140 A^2 H_1}{N} \leq \frac{150 A^2 P}{N} = O(1).$$

Since $Q \leq N^{1/2} \leq R$, one has $\frac{P}{R^2} \leq \frac{Q^2}{R^2} = O(1)$. Hence,

$$|\mathfrak{M}_1| \leq 8 \sum_{g_3^{(2)} \in \tilde{\Omega}^{(3)}} \sum_{Q_1 \leq q^{(1)}, q^{(2)} \leq Q} (q^{(1)}, q^{(2)}) \left(\frac{(y_1, P) + (y_2, P)}{R} + O(1) \right). \quad (14.52)$$

Let estimate the first summand in (14.52):

$$\frac{1}{R} \sum_{Q_1 \leq q^{(1)}, q^{(2)} \leq Q} (q^{(1)}, q^{(2)}) \sum_{y_1, y_2 < T} (y_1, P) \leq \frac{T}{R} \sum_{q^{(1)}, q^{(2)} \leq Q} (q^{(1)}, q^{(2)}) \sum_{y_1 < T} (y_1, P). \quad (14.53)$$

Because of

$$\sum_{y_1 < T} (y_1, P) \leq \sum_{\substack{d|P \\ d < T}} d \sum_{\substack{y_1 < T \\ y_1 \equiv 0 \pmod{d}}} 1 \ll_{\epsilon} TP^{\epsilon} \ll_{\epsilon} TQ^{2\epsilon},$$

we obtain

$$\frac{1}{R} \sum_{Q_1 \leq q^{(1)}, q^{(2)} \leq Q} (q^{(1)}, q^{(2)}) \sum_{y_1, y_2 < T} (y_1, P) \ll_{\epsilon} \frac{T^2 Q^{2\epsilon}}{R} \sum_{q^{(1)}, q^{(2)} \leq Q} (q^{(1)}, q^{(2)}). \quad (14.54)$$

Since

$$\sum_{q^{(1)}, q^{(2)} \leq Q} (q^{(1)}, q^{(2)}) \leq \sum_{q^{(1)} \leq Q} \sum_{d|q^{(1)}} d \sum_{\substack{q^{(2)} \leq Q \\ q^{(2)} \equiv 0 \pmod{d}}} 1 \leq 2 \sum_{q^{(1)} \leq Q} \sum_{d|q^{(1)}} Q \ll_{\epsilon} Q^{2+\epsilon}, \quad (14.55)$$

one has

$$\frac{1}{R} \sum_{Q_1 \leq q^{(1)}, q^{(2)} \leq Q} (q^{(1)}, q^{(2)}) \sum_{y_1, y_2 < T} (y_1, P) \ll_{\epsilon} \frac{T^2 Q^{2+3\epsilon}}{R}. \quad (14.56)$$

Substituting (14.56) into (14.52) and using (14.55), we obtain

$$|\mathfrak{M}_1| \ll_{\epsilon} \frac{T^2 Q^{2+\epsilon}}{R} + Q^{2+\epsilon} |\Omega^{(3)}| \ll_{\epsilon} Q^{2+\epsilon} |\Omega^{(3)}|. \quad (14.57)$$

This completes the proof of the lemma. \square

Lemma 14.7. *Under the hypotheses of Lemma 14.5 one has*

$$|\mathfrak{M}_2| \ll_{\epsilon} |\Omega^{(3)}| \left(\left(\frac{73A^2 N}{M^{(1)} \bar{K}} \right)^2 + 1 \right) N^{\epsilon} Q. \quad (14.58)$$

Proof. Since $|\mathcal{Y}| \leq \max\{x_1 y_2, x_2 y_1\} \leq x_2 y_2$, then applying the bound (14.32) to each factor we obtain

$$|\mathcal{Y}| \leq \left(\frac{73A^2 N}{M^{(1)}} \right)^2. \quad (14.59)$$

As $\mathbf{q}|\mathcal{Y}$ and $\mathcal{Y} \neq 0$, one has

$$\mathbf{q} \leq \min\{Q^2, |\mathcal{Y}|\} \leq \min\{Q^2, \left(\frac{73A^2 N}{M^{(1)}} \right)^2\} \leq \sqrt{Q^2 \left(\frac{73A^2 N}{M^{(1)}} \right)^2} = Q \frac{73A^2 N}{M^{(1)}}. \quad (14.60)$$

It follows from the conditions of the lemma and from (14.29) that

$$\mathbf{q} \frac{74A^2 \bar{K}}{M^{(1)}} \leq Q \frac{73A^2 N}{M^{(1)}} \frac{74A^2 \bar{K}}{M^{(1)}} < 1. \quad (14.61)$$

By (14.61) we obtain that the right side of (14.30) is less than $\frac{1}{\mathbf{q}}$, whence

$$\|g_3^{(1)} \frac{a^{(1)}}{q^{(1)}} - g_3^{(2)} \frac{a^{(2)}}{q^{(2)}}\|_{1,2} = 0. \quad (14.62)$$

It follows from the equation (14.62) that the congruence (14.5) holds, from which we deduced in Lemma 14.1 that

$$q^{(1)} = q^{(2)} = \mathbf{q}, \quad (g_3^{(1)} a^{(1)} - g_3^{(2)} a^{(2)})_{1,2} \equiv 0 \pmod{\mathbf{q}}. \quad (14.63)$$

Then by (14.31) we obtain

$$|g_3^{(1)} - g_3^{(2)}|_{1,2} \leq \frac{73A^2N}{M^{(1)}\overline{K}}. \quad (14.64)$$

Hence, if we fix $g_3^{(1)} \in \widetilde{\Omega}^{(3)}$, then for $g_3^{(2)}$ there will be at most $\left(\frac{73A^2N}{M^{(1)}\overline{K}}\right)^2 + 1$ choices. So we have defined \mathcal{Y} . But then there are at most $\tau(\mathcal{Y})$ choices for the number $q^{(1)} = q^{(2)} = \mathbf{q}$ since $\mathbf{q}|\mathcal{Y}$. It is known (see [16, p.91]) that

$$\tau(\mathcal{Y}) \leq (\epsilon \log 2)^{-\exp(1/\epsilon)} \mathcal{Y}^\epsilon$$

for any $\epsilon > 0$. With $q^{(1)}$ being fixed there are at most $q^{(1)} \leq Q$ choices for $a^{(1)}$. We note that if $g_3^{(1)}, g_3^{(2)}, a^{(1)}$ are fixed then $a^{(2)}$ is uniquely determined by (14.63). Actually, we write

$$g_3^{(1)} = (x_1, x_2)^t, \quad g_3^{(2)} = (y_1, y_2)^t, \quad (14.65)$$

then it follows from (14.63) that

$$x_1 a^{(1)} \equiv y_1 a^{(2)} \pmod{\mathbf{q}}, \quad x_2 a^{(1)} \equiv y_2 a^{(2)} \pmod{\mathbf{q}}. \quad (14.66)$$

Since $(a^{(1)}, \mathbf{q}) = (a^{(2)}, \mathbf{q}) = 1$, we have

$$\delta_1 = (x_1, \mathbf{q}) = (y_1, \mathbf{q}), \quad \delta_2 = (x_2, \mathbf{q}) = (y_2, \mathbf{q}), \quad (14.67)$$

and $(\delta_1, \delta_2) = 1$, as $(x_1, x_2) = 1$. Then one can obtain from (14.66) that

$$a^{(2)} \equiv A_1 a^{(1)} \pmod{\frac{\mathbf{q}}{\delta_1}}, \quad a^{(2)} \equiv A_2 a^{(1)} \pmod{\frac{\mathbf{q}}{\delta_2}}. \quad (14.68)$$

These two congruences are equivalent to the congruence modulo $[\frac{\mathbf{q}}{\delta_1}, \frac{\mathbf{q}}{\delta_2}] = \mathbf{q}$, and hence $a^{(2)}$ is uniquely determined. Finally we obtain

$$\begin{aligned} |\mathfrak{M}_2| &\leq \sum_{g_3^{(1)} \in \widetilde{\Omega}^{(3)}} \sum_{g_3^{(2)} \in \widetilde{\Omega}^{(3)}} \sum_{q^{(1)}|\mathcal{Y}} \sum_{0 < a^{(1)} \leq q^{(1)}} \mathbf{1}_{\{|g_3^{(1)} - g_3^{(2)}|_{1,2} \leq \frac{73A^2N}{M^{(1)}\overline{K}}\}} \leq \\ &\ll_\epsilon |\Omega^{(3)}| \left(\left(\frac{73A^2N}{M^{(1)}\overline{K}} \right)^2 + 1 \right) N^\epsilon Q. \end{aligned} \quad (14.69)$$

This completes the proof of the lemma. \square

Lemma 14.8. *Under the hypotheses of Lemma 14.5 one has*

$$|\mathfrak{M}_2| \ll_\epsilon |Z| \left(\left(\frac{73A^2N}{M^{(1)}} \right)^2 \frac{|\Omega^{(3)}|}{\overline{K}^2 Q_1} + \left(\frac{73A^2N}{M^{(1)}} \right)^2 Q^\epsilon + \left(\frac{73A^2N}{M^{(1)}} \right) Q^{1+\epsilon} \right). \quad (14.70)$$

Proof. Repeating the arguments in the proof of Lemma 14.7, we obtain (14.63) (14.64). To simplify we write $T = \frac{73A^2N}{M^{(1)}}$ and use the notation (14.65) and (14.67). If $g_3^{(1)}, g_3^{(2)}, a^{(1)}$ are fixed, then it has been proved in Lemma 14.7 that $a^{(2)}$ is uniquely determined by (14.63). It follows from the congruence (14.63) that $x_1y_2 \equiv x_2y_1 \pmod{\mathbf{q}}$. Thus there are $|Z|$ choices for $\frac{a^{(1)}}{q^{(1)}}$ and we obtain the inequality

$$|\mathfrak{M}_2| \leq |Z| \sum_{g_3^{(1)} \in \Omega^{(3)}} \sum_{\substack{g_3^{(2)} \in \Omega^{(3)} \\ |g_3^{(1)} - g_3^{(2)}|_{1,2} \leq \frac{T}{K}}} \mathbf{1}_{\{x_1y_2 \equiv x_2y_1 \pmod{\mathbf{q}}\}} \quad (14.71)$$

We denote

$$x_3 = \frac{x_1}{\delta_1}, x_4 = \frac{x_2}{\delta_2}, y_3 = \frac{y_1}{\delta_1}, y_4 = \frac{y_2}{\delta_2}, p = \frac{\mathbf{q}}{\delta_1\delta_2},$$

then

$$|\mathfrak{M}_2| \leq |Z| \sum_{g_3^{(1)} \in \Omega^{(3)}} \sum_{\substack{y_3 \leq \frac{T}{\delta_1} \\ |y_3 - x_3| \leq \frac{T}{K\delta_1}}} \sum_{\substack{y_4 \leq \frac{T}{\delta_2} \\ |y_4 - x_4| \leq \frac{T}{K\delta_2}}} \mathbf{1}_{\{x_3y_4 \equiv x_4y_3 \pmod{p}\}}. \quad (14.72)$$

Because of $(x_3, p) = (x_4, p) = 1$ the congruence can be written as $y_4 \equiv cy_3 \pmod{p}$, where $c \equiv x_3^{-1}x_4 \pmod{p}$. Then, using the result for the number of solutions of such congruences in [16, p.18], we obtain

$$\sum_{y_3} \sum_{y_4} \mathbf{1}_{\{x_3y_4 \equiv x_4y_3 \pmod{p}\}} = \sum_{y_3} \sum_{y_4} \mathbf{1}_{\{y_4 \equiv cy_3 \pmod{p}\}} = \frac{1}{p} \frac{T^2}{K^2\delta_1\delta_2} + O(s(\frac{c}{p}) \log^2 p), \quad (14.73)$$

where $s(\alpha) = \sum_{1 \leq i \leq s} a_i$ is the sum of partial quotients of the number $\alpha = [0; a_1, \dots, a_s]$.

Substituting (14.73) into (14.72), we have

$$|\mathfrak{M}_2| \leq |Z| \left(\frac{|\Omega^{(3)}|T^2}{K^2Q_1} + \log^2 Q \sum_{x_1 \leq T} \sum_{x_2 \leq T} s\left(\frac{x_3^{-1}x_2}{q/\delta_1}\right) \right). \quad (14.74)$$

Using the following result of Knuth and Yao [9],

$$\sum_{a \leq b} s(a/b) \ll b \log^2 b,$$

we obtain

$$\sum_{x_1 \leq T} \sum_{x_2 \leq T} s\left(\frac{x_3^{-1}x_2}{q/\delta_1}\right) \leq \sum_{x_1 \leq T} \left(\frac{T}{q/\delta_1} + 1 \right) \frac{q}{\delta_1} \log^2 q \leq (T^2 + Tq) \log^2 q. \quad (14.75)$$

Substituting (14.75) into (14.74), we obtain

$$|\mathfrak{M}_2| \ll_\epsilon |Z| \left(\frac{|\Omega^{(3)}|T^2}{K^2Q_1} + T^2Q^\epsilon + TQ^{1+\epsilon} \right). \quad (14.76)$$

This completes the proof of the lemma. \square

Corollary 14.1. *Under the hypotheses of Lemma 14.5, one has*

$$\sum_{\theta \in P_{Q_1, Q}^{(\beta)}} |S_N(\theta)|^2 \ll |\Omega_N|^2 (C_1^2 Q^{\epsilon_0} + C_2), \quad (14.77)$$

where

$$C_1 = \frac{N^{1-\delta+2\epsilon_0}}{\bar{K}Q_1^{1/2}} + N^{1-\frac{3-\alpha}{2}\delta+4,5\epsilon_0} + N^{\frac{3+\alpha}{4}-\frac{3-\alpha}{2}\delta+4,5\epsilon_0} Q^{1/2}, \quad C_2 = \frac{N^{\frac{3+\alpha}{2}-2\delta+3,5\epsilon_0} Q}{(\bar{K}Q_1)^{1-2\epsilon_0}}. \quad (14.78)$$

Proof. It was proved in Lemma 14.5 that for any $Z \subseteq P_{Q_1, Q}^{(\beta)}$ one has

$$\sum_{\theta \in Z} |S_N(\theta)| \ll |\Omega_N| |Z|^{1/2} C_1 + |\Omega_N| N^{\frac{1+\alpha}{2}-\delta+2,5\epsilon_0} Q.$$

Applying Lemma 12.5 with $W = P_{Q_1, Q}^{(\beta)}$, $f(\theta) = \frac{|S_N(\theta)|}{|\Omega_N|}$, we obtain

$$\sum_{\theta \in P_{Q_1, Q}^{(\beta)}} |S_N(\theta)|^2 \ll |\Omega_N|^2 C_1^2 Q^{\epsilon_0} + |\Omega_N| N^{\frac{1+\alpha}{2}-\delta+2,5\epsilon_0} Q \max_{\theta \in P_{Q_1, Q}^{(\beta)}} |S_N(\theta)|. \quad (14.79)$$

To estimate the maximum we apply Lemma 14.4 and obtain (14.77). This completes the proof of the corollary. \square

Corollary 14.2. *Under the hypotheses of Lemma 14.5 with $\alpha = \frac{1+\epsilon_0}{2}$, then*

$$\sum_{\substack{Q_1 \leq q \leq Q \\ 1 \leq a \leq q}}^* \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 \ll |\Omega_N|^2 \left(\frac{N^{2(1-\delta)-1/4+5\epsilon_0} Q}{(\bar{K}Q_1)^{1-2\epsilon_0}} + \frac{N^{2(1-\delta)+4\epsilon_0} Q^{1/2}}{\bar{K}^{2-2\epsilon_0} Q_1^{1-2\epsilon_0}} \right), \quad (14.80)$$

Proof. Estimating $|S_N(\frac{a}{q} + \frac{K}{N})|$ by the maximum over a, q , we obtain

$$\sum_{\substack{Q_1 \leq q \leq Q \\ 1 \leq a \leq q}}^* \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 \leq \max_{\substack{Q_1 \leq q \leq Q \\ 1 \leq a \leq q, (a, q)=1}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right| \sum_{\substack{Q_1 \leq q \leq Q \\ 1 \leq a \leq q}}^* \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|. \quad (14.81)$$

To estimate the maximum we apply Lemma 14.4. Using the first item of Lemma 14.5 with $\alpha = \frac{1+\epsilon_0}{2}$, we obtain

$$\sum_{\substack{Q_1 \leq q \leq Q \\ 1 \leq a \leq q}}^* \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 \ll |\Omega_N|^2 \left(\frac{N^{2(1-\delta)-1/4+5\epsilon_0} Q}{(\bar{K}Q_1)^{1-2\epsilon_0}} + \frac{N^{2(1-\delta)+4\epsilon_0} Q^{1/2}}{\bar{K}^{2-2\epsilon_0} Q_1^{1-2\epsilon_0}} \right).$$

This completes the proof of the corollary. \square

15 The case $\mu = 2$.

We recall that Q_0 is defined by the equation (14.9).

Lemma 15.1. *Under the hypotheses of Lemma 2.1, if the inequalities*

$$(\overline{K}Q)^{18/5+21\epsilon_0}Q < N, \quad \overline{K}Q \geq Q_0,$$

hold, then the following bound is valid

$$\sum_{\theta \in P_{Q_1, Q}^{(\beta)}} |S_N(\theta)|^2 \ll |\Omega_N|^2 \frac{\overline{K}^{\frac{36}{5}(1-\delta)} Q^{\frac{46}{5}(1-\delta)+1} \overline{K}^{34\epsilon_0} Q^{46\epsilon_0}}{(\overline{K}Q_1)^2}. \quad (15.1)$$

Proof. It follows from (14.18) that it is sufficient to prove the bound

$$\sum_{\theta \in Z^*} |S_N(\theta)|^2 \ll |\Omega_N|^2 \frac{\overline{K}^{\frac{36}{5}(1-\delta)} Q^{\frac{46}{5}(1-\delta)} \overline{K}^{34\epsilon_0} Q^{46\epsilon_0}}{(\overline{K}Q_1)^2} \quad (15.2)$$

for Z^* in (14.1). Let $Z \subseteq Z^*$ be any subset. We use the partition of the ensemble Ω_N given by the formula (12.7) (that is, by Theorem 11.4). We put $\mu = 2, \lambda = 3$ and

$$M^{(1)} = 75A^2\overline{K}Q^2, \quad M^{(2)} = (75A^2\overline{K}Q)^{13/5+11\epsilon_0}, \quad (15.3)$$

then the condition (12.45) and all conditions of Theorem 11.4 hold. Hence, all conditions of Lemma 12.4 hold and therefore the estimate (12.46) is valid. For the equation (12.44) can be written as the congruence

$$\left((g_2^{(1)} \frac{a^{(1)}}{q^{(1)}} - g_2^{(2)} \frac{a^{(2)}}{q^{(2)}}) g_3 \right)_{1,2} \equiv 0 \pmod{1}. \quad (15.4)$$

In the same way, as it was done in Lemma 14.1, we obtain

$$q^{(1)} = q^{(2)} = \mathbf{q}$$

and, hence, $a^{(1)} = a^{(2)}$, since the definition of the set Z^* in (14.1). Then the relations (12.43) and (12.44) imply that

$$\left| (g_2^{(1)} - g_2^{(2)}) g_3 \right|_{1,2} \leq \frac{73A^2N}{M^{(1)}\overline{K}}, \quad \left((g_2^{(1)} - g_2^{(2)}) g_3 \right)_{1,2} \equiv 0 \pmod{\mathbf{q}}. \quad (15.5)$$

We write

$$\eta' = g_2^{(2)} g_3, \quad \eta = g_3, \quad \gamma = g_2^{(1)}, \quad X = \|\eta'\|, \quad Y = \|g_2^{(2)}\|, \quad K_1 = \overline{K} \frac{XM^{(1)}}{73A^2N}. \quad (15.6)$$

Without loss of generality we may suppose that $\|g_2^{(1)}\| \leq \|g_2^{(2)}\|$, then $\|\gamma\| \leq Y$. It also follows from the properties of ensemble that $\|\gamma\| \asymp Y$. Moreover, it follows from the bounds proved in Theorem 11.4, that

$$\frac{\overline{K}}{(M^{(1)})^{4\epsilon_0} (M^{(2)})^{2\epsilon_0}} \ll K_1 \ll \overline{K} \frac{(M^{(2)})^{2\epsilon_0}}{(M^{(1)})^{2\epsilon_0}}, \quad \frac{(M^{(2)})}{(M^{(1)})^{2\epsilon_0}} \ll Y \ll \frac{(M^{(2)})^{1+2\epsilon_0}}{(M^{(1)})^{2\epsilon_0}}. \quad (15.7)$$

The relations (15.5) can be written as

$$|\gamma\eta - \eta'|_{1,2} < \frac{X}{K_1}, \quad (\gamma\eta - \eta')_{1,2} \equiv 0 \pmod{q}. \quad (15.8)$$

Let verify whether the conditions of Lemma 2.1 hold. To do this it is sufficient to confirm that $Y < X$, $(QK_1)^{13/5} < Y$, that is, it is enough to check that

$$(\overline{K}Q)^{13/5} < (M^{(2)})^{1-26\epsilon_0/5} (M^{(1)})^{16\epsilon_0/5}. \quad (15.9)$$

The inequality (15.9) is satisfied by the choice of parameters $M^{(1)}, M^{(2)}$. Thus to estimate the cardinality of the set $\mathfrak{M}(g_3)$ Lemma 2.1 can be applied as follows. We fix the vector $g_2^{(2)}$, for which there are $|\Omega^{(2)}|$ choices. We also fix $\frac{a^{(1)}}{q^{(1)}}$, for which there are $|Z|$ choices. It has been proved that then $\frac{a^{(2)}}{q^{(2)}}$ is fixed. Hence, for a fixed g_3 we have

$$|\mathfrak{M}(g_3)| \ll \frac{Y^2}{(K_1 Q_1)^2} |\Omega^{(2)}| |Z|. \quad (15.10)$$

Using the bounds (15.7), we obtain

$$|\mathfrak{M}(g_3)| \ll \frac{(M^{(2)})^{2+8\epsilon_0} (M^{(1)})^{4\epsilon_0}}{(\overline{K}Q_1)^2} |\Omega^{(2)}| |Z| \ll \frac{(\overline{K}Q)^{26/5+43\epsilon_0} (\overline{K}Q^2)^{4\epsilon_0}}{(\overline{K}Q_1)^2} |\Omega^{(2)}| |Z|. \quad (15.11)$$

Substituting (15.11) into (12.46), we have

$$\sum_{\theta \in Z} |S_N(\theta)| \ll |Z|^{1/2} H_1 |\Omega^{(1)}|^{1/2} |\Omega^{(3)}| |\Omega^{(2)}|^{1/2} \frac{(\overline{K}Q)^{13/5+22\epsilon_0} (\overline{K}Q^2)^{2\epsilon_0}}{\overline{K}Q_1}. \quad (15.12)$$

Using the lower bound of (11.63) and $|\Omega^{(1)}|, |\Omega^{(2)}|$, we obtain

$$\sum_{\theta \in Z} |S_N(\theta)| \ll |Z|^{1/2} |\Omega_N| \frac{(M^{(1)})^{1-\delta+2,5\epsilon_0} (\overline{K}Q)^{13/5+22\epsilon_0} (\overline{K}Q^2)^{2\epsilon_0}}{(M^{(2)})^{\delta-\epsilon_0/2} \overline{K}Q_1}. \quad (15.13)$$

Substituting (15.3) into (15.13), we have

$$\sum_{\theta \in Z} |S_N(\theta)| \ll |Z|^{1/2} |\Omega_N| \frac{\overline{K}^{\frac{18}{5}(1-\delta)} Q^{\frac{23}{5}(1-\delta)} \overline{K}^{17\epsilon_0} Q^{23\epsilon_0}}{\overline{K}Q_1}. \quad (15.14)$$

Applying Lemma 12.5 with $W = Z^*$, $c_2 = 0$, $f(\theta) = \frac{|S_N(\theta)|}{|\Omega_N|}$, we obtain (15.2). This completes the proof of the lemma. \square

Lemma 15.2. *Under the hypotheses of Lemma 2.1, if the inequality $\overline{K}q > Q_0$ holds, then the following bound is valid*

$$|S_N(\theta)| \ll \frac{|\Omega_N|}{\overline{K}q} (\overline{K}q)^{\frac{18}{5}(1-\delta)+19\epsilon_0}. \quad (15.15)$$

Proof. We use the partition of the ensemble Ω_N given by the formula (12.7) (that is, by Theorem 11.4). We put $\mu = 2$, $\lambda = 3$ and

$$M^{(1)} = 75A^2 \overline{K}q, \quad M^{(2)} = (75A^2 \overline{K}q)^{13/5+11\epsilon_0}, \quad (15.16)$$

then the condition (12.45) of Lemma 12.4 holds. Suppose that the inequality

$$(\overline{K}q)^{18/5+19\epsilon_0} < N \quad (15.17)$$

is valid. Then all conditions of Theorem 11.4 hold and, hence, all conditions of Lemma 12.4 hold. So one has the bound (12.46). Below we will use notations (15.6). Verification of the feasibility of conditions of Lemma 2.1. can be done in the same manner. Thus to estimate the cardinality of the set $\mathfrak{M}(g_3)$ Lemma 2.1 can be applied

$$|\mathfrak{M}(g_3)| \ll \frac{Y^2}{(K_1 q)^2} |\Omega^{(2)}|. \quad (15.18)$$

Using the bounds (15.7), we have

$$|\mathfrak{M}(g_3)| \ll \frac{(M^{(2)})^{2+8\epsilon_0} (M^{(1)})^{4\epsilon_0}}{(\overline{K}q)^2} |\Omega^{(2)}| \ll \frac{(\overline{K}q)^{26/5+43\epsilon_0} (\overline{K}q)^{4\epsilon_0}}{(\overline{K}q)^2} |\Omega^{(2)}|. \quad (15.19)$$

Substituting (15.19) into (12.46), we obtain

$$|S_N(\theta)| \ll H_1 |\Omega^{(1)}|^{1/2} |\Omega^{(3)}| |\Omega^{(2)}|^{1/2} (\overline{K}q)^{8/5+24\epsilon_0}. \quad (15.20)$$

Using the lower bound of (11.63) and $|\Omega^{(1)}|, |\Omega^{(2)}|$, we obtain

$$|S_N(\theta)| \ll |\Omega_N| (\overline{K}q)^{\frac{18}{5}(1-\delta)-1+19\epsilon_0}, \quad (15.21)$$

and the inequality (15.15) is proved under the condition (15.17).

Let next the inequality (15.17) be false, that is,

$$N \leq (\overline{K}q)^{18/5+19\epsilon_0}. \quad (15.22)$$

Then the conditions of Lemma 14.4 hold. Applying (14.22) and taking into account (15.22), we obtain

$$|S_N(\theta)| \ll |\Omega_N| (\overline{K}q)^{\frac{18}{5}(1-\delta)-1+9\epsilon_0}. \quad (15.23)$$

This completes the proof of the lemma. \square

16 Estimates for integrals of $|S_N(\theta)|^2$.

Lemma 16.1. *The following inequality holds*

$$\int_0^1 |S_N(\theta)|^2 d\theta \leq \frac{1}{N} \sum_{0 \leq a \leq q \leq N^{1/2}}^* \int_{|K| \leq \frac{N^{1/2}}{q}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK, \quad (16.1)$$

where \sum^* means that the sum is taken over a and q being coprime for $q \geq 1$, and $a = 0, 1$ for $q = 1$.

Proof. It follows from the Dirichlet theorem that for any $\theta \in [0, 1]$ there exist $a, q \in \mathbb{N}$ and $\beta \in \mathbb{R}$ such that

$$\theta = \frac{a}{q} + \beta, \quad (a, q) = 1, \quad 0 \leq a \leq q \leq N^{1/2}, \quad |\beta| \leq \frac{1}{qN^{1/2}},$$

so

$$\int_0^1 |S_N(\theta)|^2 d\theta = \int_0^1 \left| S_N\left(\frac{a}{q} + \beta\right) \right|^2 d\left(\frac{a}{q} + \beta\right) \leq \sum_{0 \leq a \leq q \leq N^{1/2}}^* \int_{|\beta| \leq \frac{1}{qN^{1/2}}} \left| S_N\left(\frac{a}{q} + \beta\right) \right|^2 d\beta. \quad (16.2)$$

The change of variables $j = cx + n$ in (16.2) leads to the inequality (16.1). This completes the proof of the lemma. \square

We recall that

$$Q_0 = \max \left\{ \exp \left(\frac{10^5 A^4}{\epsilon_0^2} \right), \exp(\epsilon_0^{-5}) \right\}.$$

Lemma 16.2. *The following inequality holds*

$$\begin{aligned} \int_0^1 |S_N(\theta)|^2 d\theta &\leq 2Q_0^2 \frac{|\Omega_N|^2}{N} + \frac{1}{N} \sum_{\substack{0 \leq a \leq q \leq N^{1/2} \\ q > Q_0}}^* \int_{\substack{Q_0 \leq |K| \leq \frac{N^{1/2}}{q}}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK + \\ &\frac{1}{N} \sum_{0 \leq a \leq q \leq Q_0}^* \int_{\substack{Q_0 \leq |K| \leq \frac{N^{1/2}}{q}}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK + \frac{1}{N} \sum_{\substack{1 \leq a \leq q \leq N^{1/2} \\ q > Q_0}}^* \int_{|K| \leq \frac{Q_0}{q}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK. \end{aligned} \quad (16.3)$$

Proof. To simplify we write $f(K) = \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2$, then

$$\begin{aligned} \sum_{\substack{0 \leq a \leq q \leq N^{1/2} \\ |K| \leq \frac{N^{1/2}}{q}}}^* \int f(K) dK &= \sum_{\substack{0 \leq a \leq q \leq N^{1/2} \\ q > Q_0}}^* \int_{\substack{Q_0 < |K| \leq \frac{N^{1/2}}{q}}} f(K) dK + \sum_{0 \leq a \leq q \leq Q_0}^* \int_{\substack{Q_0 < |K| \leq \frac{N^{1/2}}{q}}} f(K) dK + \\ &\sum_{\substack{1 \leq a \leq q \leq N^{1/2} \\ q > Q_0}}^* \int_{|K| \leq \frac{Q_0}{q}} f(K) dK + \sum_{1 \leq a \leq q \leq Q_0}^* \int_{|K| \leq \frac{Q_0}{q}} f(K) dK. \end{aligned} \quad (16.4)$$

We estimate the fourth integral trivially

$$\sum_{q \leq Q_0} \sum_{0 \leq a \leq q}^* \int_{|K| \leq \frac{Q_0}{q}} f(K) dK \leq 2Q_0^2 |\Omega_N|^2. \quad (16.5)$$

Substituting (16.5) into (16.4) and using (16.1), we obtain (16.3). This completes the proof of the lemma. \square

First we estimate the third integral in the right side of (16.3). It is convenient to use the following notation

$$\gamma = 1 - \delta, \quad \xi_1 = N^{2\gamma + 6\epsilon_0}. \quad (16.6)$$

Lemma 16.3. *For $\gamma < \frac{1}{8}$ and $\epsilon_0 \in (0, \frac{1}{2500})$ the following inequality holds*

$$\frac{1}{N} \sum_{\substack{1 \leq a \leq q \leq N^{1/2} \\ q > \xi_1}}^* \int_{|K| \leq \frac{Q_0}{q}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N}. \quad (16.7)$$

Proof. We denote by I the integral in the left side of (16.7). Since $\overline{K} \geq 1$, $q > Q_0$, the conditions of Lemma 14.4 hold. Applying this lemma we obtain

$$I \ll_{A, \epsilon_0} |\Omega_N|^2 \frac{N^{2\gamma+2\epsilon_0}}{q^{2-4\epsilon_0}} \int_{|K| \leq \frac{Q_0}{q}} \frac{dK}{\overline{K}^{2-4\epsilon_0}} \ll |\Omega_N|^2 \frac{N^{2\gamma+2\epsilon_0}}{q^{2-4\epsilon_0}} \frac{Q_0}{q}, \quad (16.8)$$

since $|K| \leq \frac{Q_0}{q} \leq 1$ and, hence, $\overline{K} = 1$. Substituting (16.8) into (16.7), we obtain

$$\frac{1}{N} \sum_{\substack{1 \leq a \leq q \leq N^{1/2} \\ q > \xi_1}}^* I \ll \frac{|\Omega_N|^2}{N} \sum_{\xi_1 < q \leq N^{1/2}} \frac{N^{2\gamma+2\epsilon_0}}{q^{2-4\epsilon_0}}. \quad (16.9)$$

By the choice of the parameter ξ_1 , we have

$$\sum_{\xi_1 < q \leq N^{1/2}} \frac{N^{2\gamma+2\epsilon_0}}{q^{2-4\epsilon_0}} \leq \frac{N^{2\gamma+2\epsilon_0}}{\xi_1^{1-4\epsilon_0}} \leq N^{-\epsilon_0} \ll 1. \quad (16.10)$$

Substituting (16.10) into (16.9), we obtain (16.7). This completes the proof of the lemma. \square

Lemma 16.4. For $\gamma < \frac{1}{8}$ and $\epsilon_0 \in (0, \frac{1}{2500})$ the following inequality holds

$$\frac{1}{N} \sum_{\substack{1 \leq a \leq q \leq \xi_1 \\ q > Q_0}}^* \int_{|K| \leq \frac{Q_0}{q}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N}. \quad (16.11)$$

Proof. We denote by I the integral in the left side of (16.11). Thus $q > Q_0$, so one has $\overline{K} = 1$ and, hence, all conditions of Lemma 14.3 hold. Applying this lemma we obtain

$$I \ll |\Omega_N|^2 q^{4\gamma-2+12\epsilon_0} \int_{|K| \leq \frac{Q_0}{q}} \frac{dK}{(\overline{K})^{1-4\gamma-12\epsilon_0}} \ll |\Omega_N|^2 q^{4\gamma-3+12\epsilon_0}, \quad (16.12)$$

since $|K| \leq \frac{Q_0}{q} \leq 1$ and, hence, $\overline{K} = 1$. Substituting (16.12) into (16.11), we have

$$\frac{1}{N} \sum_{\substack{1 \leq a \leq q \leq \xi_1 \\ q > Q_0}}^* I \ll \frac{|\Omega_N|^2}{N} \sum_{1 \leq q \leq \xi_1} q^{4\gamma-2+12\epsilon_0}. \quad (16.13)$$

By the choice of parameters δ, ϵ_0 we obtain (16.11). This completes the proof of the lemma. \square

Thus we have estimated the third integral in the right side of (16.3). Next we estimate the second one.

Lemma 16.5. Under the hypotheses of Lemma 2.1 for $\gamma < \frac{1}{8}$ and $\epsilon_0 \in (0, \frac{1}{2500})$ the following inequality holds

$$\frac{1}{N} \sum_{0 \leq a \leq q \leq Q_0}^* \int_{\substack{\frac{Q_0}{q} \leq |K| \leq \frac{N^{1/2}}{q}}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N}. \quad (16.14)$$

Proof. We denote by I the integral in the left side of (16.14). Applying Lemma 15.2, we obtain

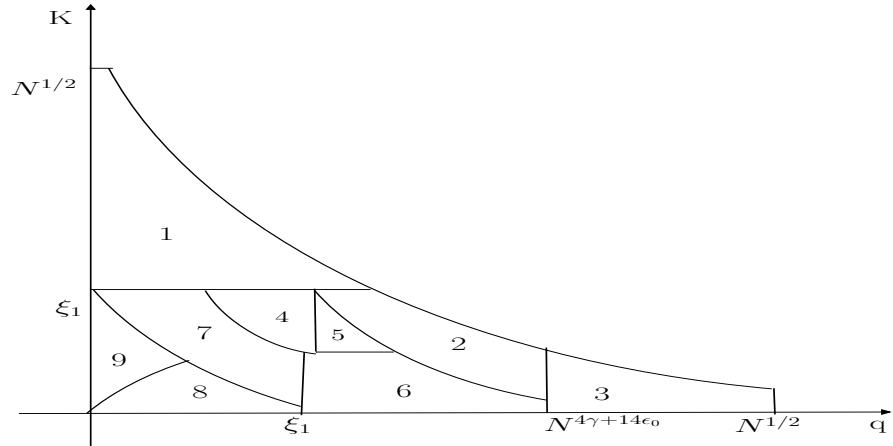
$$I \ll |\Omega_N|^2 q^{\frac{36}{5}\gamma-2+38\epsilon_0} \int_{\frac{Q_0}{q} \leq |K| \leq \frac{N^{1/2}}{q}} (\overline{K})^{\frac{36}{5}\gamma-2+38\epsilon_0} dK \ll \frac{|\Omega_N|^2}{q}, \quad (16.15)$$

by the choice of the parameter γ . Summing (16.15) over $0 \leq a \leq q \leq Q_0$, we obtain (16.14). This completes the proof of the lemma. \square

It remains to estimate the first integral in the right side of (16.3), that is,

$$\frac{1}{N} \sum_{\substack{0 \leq a \leq q \leq N^{1/2} \\ q > Q_0}}^* \int_{\frac{Q_0}{q} \leq |K| \leq \frac{N^{1/2}}{q}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK. \quad (16.16)$$

The following lemmas will be devoted to this. We partition the range of summation and integration over q , K into nine subareas:



Lemma 16.6 corresponds to the domain 1, Lemma 16.8 corresponds to the domain 2, Lemma 16.9 corresponds to the domain 3, Lemma 16.10 corresponds to the domain 4, Lemma 16.11 corresponds to the domain 5, Lemma 16.12 corresponds to the domain 6, Lemma 16.13 corresponds to the domain 7, Lemma 16.14 corresponds to the domain 8, Lemma 16.15 corresponds to the domain 9.

Lemma 16.6. For $\gamma \leq \frac{13-\sqrt{145}}{8} - 5\epsilon_0$, $\epsilon_0 \in (0, \frac{1}{2500})$ the following inequality holds

$$\frac{1}{N} \sum_{\substack{1 \leq a \leq q \leq N^{1/2} \\ q > Q_0}}^* \int_{\xi_1 \leq |K| \leq \frac{N^{1/2}}{q}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N}. \quad (16.17)$$

Proof. We denote by I the integral in the left side of (16.17). Applying Lemma 14.4, we obtain

$$I \ll |\Omega_N|^2 \frac{N^{2\gamma+2\epsilon_0}}{q^{2-4\epsilon_0}} \int_{\xi_1 \leq |K| \leq \frac{N^{1/2}}{q}} \frac{dK}{K^{2-4\epsilon_0}} \ll |\Omega_N|^2 \frac{N^{2\gamma+2\epsilon_0}}{q^{2-4\epsilon_0} \xi_1^{1-4\epsilon_0}}. \quad (16.18)$$

Substituting (16.18) into the left side of (16.17), we have

$$\frac{1}{N} \sum_{\substack{1 \leq a \leq q \leq N^{1/2} \\ q > Q_0}}^* \int_{\xi_1 \leq |K| \leq \frac{N^{1/2}}{q}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N} \sum_{Q_0 < q \leq N^{1/2}} \frac{N^{2\gamma+2\epsilon_0}}{q^{1-4\epsilon_0} \xi_1^{1-4\epsilon_0}}. \quad (16.19)$$

By the choice of the parameter ξ_1 , we obtain

$$\sum_{Q_0 < q \leq N^{1/2}} \frac{N^{2\gamma+2\epsilon_0}}{q^{1-4\epsilon_0} \xi_1^{1-4\epsilon_0}} \ll \frac{N^{2\gamma+4\epsilon_0}}{\xi_1^{1-4\epsilon_0}} \ll N^{-\epsilon_0/2} \ll 1. \quad (16.20)$$

Substituting (16.20) into (16.19), we obtain (16.17). This completes the proof of the lemma. \square

Let

$$c_1 = c_1(N), c_2 = c_2(N), Q_0 \leq c_1 < c_2 \leq N^{1/2},$$

and let

$$f_1 = f_1(N, q), f_2 = f_2(N, q), \frac{Q_0}{q} \leq f_1 < f_2 \leq \frac{N^{1/2}}{q},$$

$$m_1 = \min\{f_1(N, N_j), f_1(N, N_{j+1})\}, m_2 = \max\{f_2(N, N_j), f_2(N, N_{j+1})\}.$$

We recall that the sequence $\{N_j\}_{j=-1}^{J+1}$ was defined in (9.4).

Lemma 16.7. If the functions $f_1(N, q), f_2(N, q)$ are monotonic for q , then the following inequality holds

$$\sum_{\substack{c_1 \leq q \leq c_2 \\ 1 \leq a \leq q}}^* \int_{f_1 \leq |K| \leq f_2} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \leq$$

$$\leq \sum_{j: c_1^{1-\epsilon_0} \leq N_j \leq c_2} \int_{m_1 \leq |K| \leq m_2} \sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK. \quad (16.21)$$

Proof. The interval $[c_1, c_2]$ can be covered by the intervals $[N_j, N_{j+1}]$, then

$$\sum_{c_1 \leq q \leq c_2} \leq \sum_{j: c_1^{1-\epsilon_0} \leq N_j \leq c_2} \sum_{N_j \leq q \leq N_{j+1}}.$$

Interchanging the order of summation over q and integration over K , we obtain (16.21). This completes the proof of the lemma. \square

To simplify we denote $Q_1 = N_j, Q = N_{j+1}$. Using the relation of Lemma 9.1, we obtain

$$\frac{Q}{Q_1} \leq Q^{\epsilon_0} \leq N^{\epsilon_0/2}, \quad c_1^{1-\epsilon_0} \leq Q_1 \leq c_2, \quad c_1 \leq Q \leq c_2^{1+2\epsilon_0}.$$

Further, we will use this bounds without reference to them. We recall that $\overline{K} = \max\{1, |K|\}$. We note that we will always have $m_2 \geq 1$, thus for $\eta < 1$ we have

$$\int_{m_1 \leq |K| \leq m_2} \frac{dK}{\overline{K}^\eta} \ll m_2^{1-\eta}. \quad (16.22)$$

For $\eta > 1$ we always have

$$\int_{m_1 \leq |K| \leq m_2} \frac{dK}{\overline{K}^\eta} < \frac{1}{m_1^{\eta-1}}. \quad (16.23)$$

However, if $m_1 \leq 1$, then for $\eta > 1$ one has

$$\int_{m_1 \leq |K| \leq m_2} \frac{dK}{\overline{K}^\eta} \ll 1. \quad (16.24)$$

Lemma 16.8. For $\gamma \leq \frac{13-\sqrt{145}}{8} - 5\epsilon_0$, $\epsilon_0 \in (0, \frac{1}{2500})$ the following inequality holds

$$\frac{1}{N} \sum_{\substack{1 \leq a \leq q \leq N^{4\gamma+14\epsilon_0} \\ q > N^{2\gamma+9\epsilon_0}}}^* \int_{\substack{N^{4\gamma+15\epsilon_0} \\ q}}^{\substack{N^{1/2} \\ q}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N}. \quad (16.25)$$

Proof. We use Lemma 16.7 with

$$c_1 = N^{2\gamma+9\epsilon_0}, \quad c_2 = N^{4\gamma+14\epsilon_0}, \quad f_1 = \frac{N^{4\gamma+15\epsilon_0}}{q}, \quad f_2 = \frac{N^{1/2}}{q}, \quad m_1 = \frac{N^{4\gamma+15\epsilon_0}}{Q}, \quad m_2 = \frac{N^{1/2}}{Q_1}.$$

We note that $KQ \leq N^{1/2} \frac{Q}{Q_1} \leq N^{1/2+\epsilon_0/2}$, thus, applying Corollary 14.2, we obtain

$$\sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 \ll |\Omega_N|^2 \left(\frac{N^{2\gamma-1/4+5\epsilon_0} Q}{(\overline{K} Q_1)^{1-2\epsilon_0}} + \frac{N^{2\gamma+4\epsilon_0} Q^{1/2}}{\overline{K}^{2-2\epsilon_0} Q_1^{1-2\epsilon_0}} \right). \quad (16.26)$$

Using (16.22) while integrating the first summand over K and using (16.23) while integrating the second one, we obtain

$$\int_{m_1 \leq |K| \leq m_2} \sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll |\Omega_N|^2 \left(N^{2\gamma-\frac{1}{4}+7\epsilon_0} \frac{Q}{Q_1} + N^{2\gamma+4\epsilon_0+(4\gamma+15\epsilon_0)(-1+2\epsilon_0)} \frac{Q^{3/2}}{Q_1} \right).$$

Thus $Q \leq N^{4\gamma+15\epsilon_0}$, $\frac{Q}{Q_1} \leq N^{\epsilon_0/2}$ and $\gamma < \frac{1}{8} - 4\epsilon_0$, we obtain

$$\int_{m_1 \leq |K| \leq m_2} \sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll |\Omega_N|^2 N^{-0,1\epsilon_0}. \quad (16.27)$$

Since the number of summands in the sum over j is less than $\log \log N$, one has

$$\sum_{j: c_1^{1-\epsilon_0} \leq N_j \leq c_2 m_1 \leq |K| \leq m_2} \int \sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll |\Omega_N|^2.$$

This completes the proof of the lemma. \square

Lemma 16.9. For $\gamma \leq \frac{13-\sqrt{145}}{8} - 5\epsilon_0$, $\epsilon_0 \in (0, \frac{1}{2500})$ the following inequality holds

$$\frac{1}{N} \sum_{\substack{1 \leq a \leq q \leq N^{1/2} \\ q > N^{4\gamma+14\epsilon_0}}}^* \int_{\substack{Q_0 \leq |K| \leq \frac{N^{1/2}}{q}}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N}. \quad (16.28)$$

Proof. We use Lemma 16.7 with

$$c_1 = N^{4\gamma+14\epsilon_0}, c_2 = N^{1/2}, f_1 = \frac{Q_0}{q}, f_2 = \frac{N^{1/2}}{q}, m_1 = \frac{Q_0}{Q}, m_2 = \frac{N^{1/2}}{Q_1}.$$

Applying Corollary 14.2 we obtain the bound (16.26). Using (16.22) while integrating the first summand over K and using (16.24) while integrating the second one, we obtain

$$\int_{m_1 \leq |K| \leq m_2} \sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll |\Omega_N|^2 \left(N^{2\gamma-1/4+7\epsilon_0} \frac{Q}{Q_1} + N^{2\gamma+4\epsilon_0} \frac{Q}{Q_1^{1-2\epsilon_0} Q^{1/2}} \right).$$

Thus $\frac{Q}{Q_1^{1-2\epsilon_0}} \leq N^{1,5\epsilon_0}$, $Q^{1/2} \geq N^{2\gamma+7\epsilon_0}$ and $\gamma < \frac{1}{8} - 4\epsilon_0$, we have

$$\int_{m_1 \leq |K| \leq m_2} \sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll |\Omega_N|^2 N^{-\epsilon_0}.$$

Since the number of summands in the sum over j is less than $\log \log N$, the lemma is proved. \square

Lemma 16.10. For $\gamma \leq \frac{13-\sqrt{145}}{8} - 5\epsilon_0$, $\epsilon_0 \in (0, \frac{1}{2500})$ the following inequality holds

$$\frac{1}{N} \sum_{\substack{1 \leq a \leq q \leq N^{2\gamma+9\epsilon_0} \\ q > N^{\gamma+6\epsilon_0}}}^* \int_{\substack{N^{3\gamma+12\epsilon_0} \leq |K| \leq \xi_1}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N}. \quad (16.29)$$

Proof. We use Lemma 16.7 with

$$c_1 = N^{\gamma+6\epsilon_0}, c_2 = N^{2\gamma+9\epsilon_0}, f_1 = \frac{N^{3\gamma+12\epsilon_0}}{q}, f_2 = \xi_1, m_1 = \frac{N^{3\gamma+12\epsilon_0}}{Q}, m_2 = \xi_1.$$

Applying Corollary 14.2 we obtain the bound (16.26). Using (16.22) while integrating the first summand over K and using (16.23) while integrating the second one, we obtain

$$\int_{m_1 \leq |K| \leq m_2} \sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll |\Omega_N|^2 \left(N^{2\gamma-1/4+7\epsilon_0} + N^{2\gamma+5\epsilon_0+(3\gamma+12\epsilon_0)(-1+2\epsilon_0)} \frac{Q^{3/2}}{Q_1} \right).$$

Thus $\frac{Q}{Q_1} \leq Q^{\epsilon_0}$, $Q \leq N^{2\gamma+10\epsilon_0}$ and $\gamma < \frac{1}{8} - 4\epsilon_0$, we obtain

$$\int_{m_1 \leq |K| \leq m_2} \sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll |\Omega_N|^2 N^{-0,1\epsilon_0}. \quad (16.30)$$

Since the number of summands in the sum over j is less than $\log \log N$, the lemma is proved. \square

Lemma 16.11. For $\gamma \leq \frac{13-\sqrt{145}}{8} - 5\epsilon_0$, $\epsilon_0 \in (0, \frac{1}{2500})$ the following inequality holds

$$\frac{1}{N} \sum_{\substack{1 \leq a \leq q \leq N^{3\gamma+11\epsilon_0} \\ q > N^{2\gamma+9\epsilon_0}}}^* \int_{N^{\gamma+4\epsilon_0} \leq |K| \leq \frac{N^{4\gamma+15\epsilon_0}}{q}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N}. \quad (16.31)$$

Proof. We use Lemma 16.7 with

$$c_1 = N^{2\gamma+9\epsilon_0}, c_2 = N^{3\gamma+11\epsilon_0}, f_1 = N^{\gamma+4\epsilon_0}, f_2 = \frac{N^{4\gamma+15\epsilon_0}}{q}, m_1 = N^{\gamma+4\epsilon_0}, m_2 = \frac{N^{4\gamma+15\epsilon_0}}{Q_1}.$$

Applying Corollary 14.2 we obtain the bound (16.26). Using (16.22) while integrating the first summand over K and using (16.23) while integrating the second one, we obtain

$$\int_{m_1 \leq |K| \leq m_2} \sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll |\Omega_N|^2 \left(N^{2\gamma-1/4+7\epsilon_0} + N^{2\gamma+6,5\epsilon_0+(\gamma+4\epsilon_0)(-1+2\epsilon_0)} \frac{1}{Q^{1/2}} \right).$$

Thus $Q \geq N^{\gamma+9\epsilon_0}$ and $\gamma < \frac{1}{8} - 4\epsilon_0$, we obtain

$$\int_{m_1 \leq |K| \leq m_2} \sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll |\Omega_N|^2 N^{-0,1\epsilon_0}. \quad (16.32)$$

Since the number of summands in the sum over j is less than $\log \log N$, the lemma is proved. \square

Lemma 16.12. For $\gamma \leq \frac{13-\sqrt{145}}{8} - 5\epsilon_0$, $\epsilon_0 \in (0, \frac{1}{2500})$ the following inequality holds

$$\frac{1}{N} \sum_{\substack{1 \leq a \leq q \leq N^{4\gamma+14\epsilon_0} \\ q > \xi_1}}^* \int_{\substack{Q_0 \leq |K| \leq \min\{N^{\gamma+6\epsilon_0}, \frac{N^{4\gamma+15\epsilon_0}}{q}\}}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N}. \quad (16.33)$$

Proof. We use Lemma 16.7 with

$$c_1 = \xi_1, c_2 = N^{4\gamma+14\epsilon_0}, f_1 = \frac{Q_0}{q}, f_2 = \min\{N^{\gamma+6\epsilon_0}, \frac{N^{4\gamma+15\epsilon_0}}{q}\}, m_1 = \frac{Q_0}{Q}, m_2 = \min\{N^{\gamma+6\epsilon_0}, \frac{N^{4\gamma+15\epsilon_0}}{Q_1}\}$$

and estimate the inner sum in the obtained relation. Applying Corollary 14.1 with $\alpha = 4\gamma + 15, 5\epsilon_0$, we obtain

$$\sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 \ll |\Omega_N|^2 (C_1^2 Q^{\epsilon_0} + C_2), \quad (16.34)$$

where

$$C_1^2 \leq \frac{N^{2\gamma+4\epsilon_0}}{K^2 Q_1} + N^{-4\gamma^2+7\gamma-1+25\epsilon_0} + N^{-4\gamma^2+9\gamma-\frac{3}{2}+33\epsilon_0} Q, \quad C_2 = \frac{N^{4\gamma-\frac{1}{2}+12\epsilon_0} Q}{(K Q_1)^{1-2\epsilon_0}}, \quad (16.35)$$

Since the number of summands in the sum over j is less than $\log \log N$, to prove the lemma it is sufficient after taking an integral to obtain a quantity less than $N^{-0,1\epsilon_0}$. We estimate separately the integral of both summands in (16.34). We note that for $\eta \leq 1$ one has

$$\int_{m_1 \leq |K| \leq m_2} Q_1^\eta dK < Q_1^\eta \min\{N^{\gamma+6\epsilon_0}, \frac{N^{4\gamma+15\epsilon_0}}{Q_1}\} \leq Q_1^\eta N^{(\gamma+6\epsilon_0)(1-\eta)} \left(\frac{N^{4\gamma+15\epsilon_0}}{Q_1} \right)^\eta = N^{(3\gamma+9\epsilon_0)\eta+\gamma+6\epsilon_0}. \quad (16.36)$$

1. The estimate of $C_1^2 Q^{\epsilon_0}$. Using (16.24) while integrating the first summand over K and using (16.36) while integrating the second and the third one, we obtain

$$\begin{aligned} \int_{m_1 \leq |K| \leq m_2} C_1^2 Q^{\epsilon_0} dK &\ll \frac{N^{2\gamma+4\epsilon_0} Q^{\epsilon_0}}{Q_1} + N^{-4\gamma^2+8\gamma-1+31\epsilon_0} + \\ &+ N^{-4\gamma^2+9\gamma-\frac{3}{2}+33\epsilon_0} N^{4\gamma+16\epsilon_0}. \end{aligned} \quad (16.37)$$

Since $Q > \xi_1$, and substituting ξ_1 in the first summand we obtain

$$\frac{N^{2\gamma+4\epsilon_0} Q^{\epsilon_0}}{Q_1} < \frac{N^{2\gamma+4\epsilon_0} Q^{2\epsilon_0}}{Q} < \frac{N^{2\gamma+5\epsilon_0}}{N^{2\gamma+6\epsilon_0}} < N^{-0,1\epsilon_0}.$$

In view of the conditions on γ, ϵ_0 the second and the third summands are less than $N^{-0,1\epsilon_0}$.

2. The estimate of C_2 . Using (16.24) while integrating over K we obtain

$$\int_{m_1 \leq |K| \leq m_2} C_2 dK \ll N^{4\gamma-\frac{1}{2}+12\epsilon_0} \frac{Q}{Q_1} N^{\epsilon_0} \leq N^{4\gamma-\frac{1}{2}+14\epsilon_0} < N^{-0,1\epsilon_0}.$$

The last inequality holds in view of the conditions on γ, ϵ_0 .

This completes the proof of the lemma. \square

Lemma 16.13. For $\gamma \leq \frac{13-\sqrt{145}}{8} - 5\epsilon_0$, $\epsilon_0 \in (0, \frac{1}{2500})$ the following inequality holds

$$\frac{1}{N} \sum_{\substack{1 \leq a \leq q \leq \xi_1 \\ q > Q_0}}^* \int_{\substack{\xi_1 \leq |K| \leq \min\{\xi_1, \frac{N^{3\gamma+12\epsilon_0}}{q}\}}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N}. \quad (16.38)$$

Proof. We use Lemma 16.7 with

$$c_1 = Q_0, c_2 = \xi_1, f_1 = \frac{\xi_1}{q}, f_2 = \min\left\{\xi_1, \frac{N^{3\gamma+12\epsilon_0}}{q}\right\}, m_1 = \frac{\xi_1}{Q}, m_2 = \min\left\{\xi_1, \frac{N^{3\gamma+12\epsilon_0}}{Q_1}\right\}.$$

and estimate the inner sum in the obtained relation. Applying Corollary 14.1 with $\alpha = 3\gamma + 12, 5\epsilon_0$, we obtain

$$\sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 \ll |\Omega_N|^2 (C_1^2 Q^{\epsilon_0} + C_2), \quad (16.39)$$

where

$$C_1^2 \leq \frac{N^{2\gamma+4\epsilon_0}}{\bar{K}^2 Q_1} + N^{-3\gamma^2+6\gamma-1+22\epsilon_0} + N^{-3\gamma^2+7,5\gamma-1,5+28\epsilon_0} Q, \quad C_2 = \frac{N^{3,5\gamma-\frac{1}{2}+10\epsilon_0} Q}{(\bar{K} Q_1)^{1-2\epsilon_0}}. \quad (16.40)$$

As usual it is sufficient to obtain a quantity less than $N^{-0,1\epsilon_0}$ after taking an integral over K . The integral of C_2 can be estimated in the same way as in Lemma 16.12. Thus we have to estimate only the integral of $C_1^2 Q^{\epsilon_0}$. We note that for $\eta \leq 1$ one has

$$\int_{m_1 \leq |K| \leq m_2} Q_1^\eta dK < Q_1^\eta \min\left\{\xi_1, \frac{N^{3\gamma+12\epsilon_0}}{Q_1}\right\} \leq Q_1^\eta \xi_1^{(1-\eta)} \left(\frac{N^{3\gamma+12\epsilon_0}}{Q_1}\right)^\eta = N^{(\gamma+6\epsilon_0)\eta+2\gamma+6\epsilon_0}. \quad (16.41)$$

We estimate separately the integral of each of the three summands in $C_1^2 Q^{\epsilon_0}$. Using (16.23) while integrating the first summand over K and using (16.41) while integrating the second and the third one, we obtain

$$\int_{m_1 \leq |K| \leq m_2} C_1^2 Q^{\epsilon_0} dK \ll \frac{N^{2\gamma+4\epsilon_0} Q^{\epsilon_0}}{Q_1} \frac{Q}{\xi_1} + N^{-3\gamma^2+8\gamma-1+28\epsilon_0} + N^{-3\gamma^2+10,5\gamma-1,5+41\epsilon_0} \xi_1 Q.$$

The first summand is less than $N^{-0,1\epsilon_0}$ since the definition of ξ_1 . The second and the third summand are less than $N^{-0,1\epsilon_0}$ because of the conditions on γ, ϵ_0 . This completes the proof of the lemma. \square

We denote

$$\nu = \frac{\sqrt{369} - 7}{20} = 0,61\dots, \quad Q_C = \xi_1^{\frac{1}{\nu+1}}. \quad (16.42)$$

Lemma 16.14. For $\gamma \leq \frac{5}{\sqrt{369}+23} - 8\epsilon_0$, $\epsilon_0 \in (0, \frac{1}{2500})$ the following inequality holds

$$\frac{1}{N} \sum_{\substack{1 \leq a \leq q \leq \xi_1 \\ q > Q_0}}^* \int_{\substack{Q_0 \leq |K| \leq \min\{q^\nu, \frac{\xi_1}{q}\}}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N}. \quad (16.43)$$

Proof. We partition the left side of (16.43) into two sums

$$\sum_{\substack{1 \leq a \leq q \leq \xi_1 \\ q > Q_0}}^* \int_{\substack{Q_0 \leq |K| \leq \min\{q^\nu, \frac{\xi_1}{q}\}}} = \sum_{\substack{1 \leq a \leq q \leq Q_C \\ q > Q_0}}^* \int_{\substack{Q_0 \leq |K| \leq q^\nu}} + \sum_{\substack{1 \leq a \leq q \leq \xi_1 \\ q > Q_C}}^* \int_{\substack{Q_0 \leq |K| \leq \frac{\xi_1}{q}}} \quad (16.44)$$

For both of these sums we apply Lemma 16.7 with

$$c_1 = Q_0, c_2 = Q_C, f_1 = \frac{Q_0}{q}, f_2 = q^\nu, m_1 = \frac{Q_0}{Q_1}, m_2 = Q^\nu$$

for the first sum and with

$$c_1 = Q_C, c_2 = \xi_2, f_1 = \frac{Q_0}{q}, f_2 = \frac{\xi_1}{q}, m_1 = \frac{Q_0}{Q_1}, m_2 = \frac{\xi_1}{Q_1}$$

for the second one. To estimate both sums we use Lemma 14.2:

$$\sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 \ll |\Omega_N|^2 \frac{\bar{K}^{4\gamma+12\epsilon_0} Q^{6\gamma+1+20\epsilon_0}}{\bar{K} Q_1^2}. \quad (16.45)$$

The conditions of Lemma 14.2 hold in both domains because of the choice of the parameter ξ_1 and the conditions on γ, ϵ_0 :

$$\bar{K}^2 Q^3 \leq Q_C^{3+2\nu+5\epsilon_0} \leq N^{1-1,5\epsilon_0}, \quad \bar{K}^2 Q^3 \leq \xi_1^2 Q^{1+2\epsilon_0} \leq \xi_1^{3+5\epsilon_0} \leq N^{1-1,5\epsilon_0}.$$

Using (16.22) while integrating the first summand over K we obtain

$$\int_{m_1 \leq |K| \leq m_2} \sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 \ll |\Omega_N|^2 Q^{6\gamma-1+22\epsilon_0} Q^{\nu(4\gamma+12\epsilon_0)}. \quad (16.46)$$

For the sum over j to be bounded by a constant, it is sufficient to have

$$\gamma \leq \frac{1 - (23 + 12\nu)\epsilon_0}{6 + 4\nu} \Rightarrow 6\gamma - 1 + 22\epsilon_0 + \nu(4\gamma + 12\epsilon_0) \leq -0, 1\epsilon_0. \quad (16.47)$$

Substituting the value of ν , we obtain that it is sufficient to have $\gamma \leq \frac{5}{\sqrt{369+23}} - 5\epsilon_0$. Using (16.22) while integrating over K , we obtain for the second sum

$$\begin{aligned} \int_{m_1 \leq |K| \leq m_2} \sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 &\ll |\Omega_N|^2 Q^{6\gamma-1+22\epsilon_0} \frac{\xi_1^{4\gamma+12\epsilon_0}}{Q^{4\gamma+12\epsilon_0}} \leq \\ &\leq Q_C^{2\gamma-1+10\epsilon_0} \xi_1^{4\gamma+12\epsilon_0} = \xi_1^{4\gamma+12\epsilon_0+(2\gamma-1+10\epsilon_0)/(\nu+1)} \end{aligned}$$

For the sum over j to be bounded by a constant, it is sufficient to have

$$4\gamma + 12\epsilon_0 + (2\gamma - 1 + 10\epsilon_0)/(\nu + 1) < -0, 1\epsilon_0,$$

which is equivalent to

$$\gamma \leq \frac{1 - (23 + 12\nu)\epsilon_0}{6 + 4\nu}.$$

This inequality can be checked in the same manner as (16.47). This completes the proof of the lemma. \square

Lemma 16.15. For $\gamma \leq \frac{5}{\sqrt{369+23}} - 8\epsilon_0$, $\epsilon_0 \in (0, \frac{1}{2500})$ the following inequality holds

$$\frac{1}{N} \sum_{\substack{1 \leq a \leq q \leq Q_C \\ q > Q_0}}^* \int_{q^\nu \leq |K| \leq \frac{\xi_1}{q}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N}. \quad (16.48)$$

Proof. We use Lemma 16.7 with

$$c_1 = Q_0, c_2 = Q_C, f_1 = q^\nu, f_2 = \frac{\xi_1}{q}, m_1 = Q_1^\nu, m_2 = \frac{\xi_1}{Q_1}$$

Let verify the conditions of Lemma 15.1:

$$(\overline{K}Q)^{18/5+21\epsilon_0} Q \leq \left(\frac{\xi_1 Q}{Q_1} \right)^{18/5+21\epsilon_0} Q \leq \xi_1^{18/5+21\epsilon_0} Q_C^{1+5\epsilon_0} \leq \xi_1^{\frac{18\nu+23}{5(\nu+1)}+24,5\epsilon_0} < N$$

Substituting ξ_1 , we obtain that the last inequality follows from

$$\gamma \leq \frac{5(1+\nu)}{46+36\nu} - 5\epsilon_0,$$

which holds in view of the definition of ν and the conditions on γ . Applying Lemma 15.1, we obtain

$$\sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 \ll |\Omega_N|^2 \overline{K}^{\frac{36}{5}\gamma+34\epsilon_0-2} Q^{\frac{46}{5}\gamma+48\epsilon_0-1}. \quad (16.49)$$

Using (16.23) while integrating over K , we obtain

$$\begin{aligned} \int_{m_1 \leq |K| \leq m_2} \sum_{\substack{N_j \leq q \leq N_{j+1} \\ 1 \leq a \leq q}}^* \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK &\ll |\Omega_N|^2 Q^{\frac{46}{5}\gamma+48\epsilon_0-1} Q_1^{\nu(\frac{36}{5}\gamma+34\epsilon_0-1)} \leq \\ &\leq |\Omega_N|^2 Q^{\frac{46}{5}\gamma-1+\nu(\frac{36}{5}\gamma-1)+72\epsilon_0} \end{aligned}$$

For the sum over j to be bounded by a constant, it is sufficient to have

$$\gamma \leq \frac{5(1+\nu)}{46+36\nu} - 6\epsilon_0.$$

This inequality holds in view of the definition of ν and the conditions on γ . This completes the proof of the lemma. \square

17 The proof of Theorem 2.1.

Let $\gamma < \frac{5}{\sqrt{369+23}}$. We choose ϵ_0 such that $\epsilon_0 \in (0, \frac{1}{2500})$ and $\gamma \leq \frac{5}{\sqrt{369+23}} - 8\epsilon_0$. Then it follows from Lemma 16.6 –16.15 that the first integral in the right side of (16.3) is less than $\frac{|\Omega_N|^2}{N}$, that is,

$$\frac{1}{N} \sum_{\substack{0 \leq a \leq q \leq N^{1/2} \\ q > Q_0}}^* \int_{\frac{Q_0}{q} \leq |K| \leq \frac{N^{1/2}}{q}} \left| S_N\left(\frac{a}{q} + \frac{K}{N}\right) \right|^2 dK \ll \frac{|\Omega_N|^2}{N}. \quad (17.1)$$

. Substituting (17.1) and the results of Lemma 16.3 –16.5 in Lemma 16.2, we obtain

$$\int_0^1 |S_N(\theta)|^2 d\theta \ll \frac{|\Omega_N|^2}{N} \quad \text{for } \gamma < \frac{27 - \sqrt{633}}{16}. \quad (17.2)$$

Thus the inequality (7.5) is proved. This, as it was proved in the section §7, is enough for proving Theorem 2.1. This completes the proof of the theorem.

Remark 17.1. *As mentioned in Remark 2.2 the condition (2.1) can be replaced by*

$$(qK)^{\frac{64}{25} + 4\epsilon_0} < Y < X. \quad (17.3)$$

After the necessary changes of Lemma 15.1 we obtain that the optimal value of the parameter ν is $\nu = \frac{\sqrt{2274}-18}{50} = 0,59\dots$. Hence, the statements of Lemma 16.14, (16.15) hold for $\gamma \leq \frac{25}{114+2\sqrt{2274}} - 10\epsilon_0$, $\epsilon_0 \in (0, \frac{1}{2500})$. Then it follows from Lemma 16.6 –16.15 that the statement of Theorem 2.1 hold for

$$\delta_A > 1 - \frac{25}{114 + 2\sqrt{2274}} = 0,8805\dots \quad (17.4)$$

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